NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 2209

DEPARTMENT OF APPLIED MATHEMATICS

SMA2209 MATHEMATICAL MODELLING

MAY 2003

Time : 3 hours

This paper contains TWO sections. Answer ALL questions in section A and TWO questions from section B.

SECTION A: Answer ALL questions in this section [40].

A1. Write down the major factors that must be considered when constructing a mathematical model. [3]

A2. Rival supermarkets enter into a price war for dog food. Supermarket A drops its price and as a result the daily sales volume of supermarket B is changed by an amount proportional to:

(a) The difference between B's price and A's price.

(b) The difference between B's price and the recommended retail price.

(c) The difference between A's price and the recommended retail price.

Construct a model for the new sales volume of supermarket B. [5]

A3. Consider a chemical which can diffuse between three identical compartments. The chemical flows from one compartment to an adjacent one at a rate which is proportional to the difference in the concentrations in the compartments and towards the compartment with the lower concentration. In addition the chemical leaks out of each compartment at a rate proportional to its concentration in the compartment. If the concentrations of the chemical in the three compartments are c_1, c_2 and c_3 the equations for this system are:

$$\begin{aligned}
\dot{c}_1 &= -\alpha(c_1 - c_2) - \gamma c_1 \\
\dot{c}_2 &= \alpha(c_1 - c_2) - \alpha(c_2 - c_3) - \gamma c_2 \\
\dot{c}_1 &= \alpha(c_2 - c_3) - \gamma c_3
\end{aligned}$$

where α and γ are strictly positive diffusion constants.

Draw a compartmentalized diagram for the above system. [5]

A4. At a school with N pupils it was found out that 10 pupils were infected with smallpox. It was assumed that the rate of change in the infected pupils is proportional to the product of the number of pupils who have the disease with the number that are disease free.

(a) By letting $I(t)$ denote the number of pupils with smallpox at time, t weeks show that

$$\frac{dI}{dt} = kI(N - I)$$

where k is a positive constant. [2]

(b) Find the general solution of the equation in (a). [5]

(c) Show that as $t \to \infty$ the infected population I converges to N. [3]

(d) Given that the school had 1000 pupils how long will it take for a quarter of the pupils to contract the disease, given that after five weeks 15 pupils had contracted the disease? [6]

A5. (a) The period, T, of the oscillations depends only on the mass m, the length l of the thread and the acceleration, g of free fall at the place concerned. Suppose then that

$$T = km^a l^b g^c$$

where a, b, c and k are unknown numbers, use dimensional analysis to find the relationship between T, m, l and g. [3]

(b) Check whether the following equation is dimensionally consistent:

$$\frac{dp}{dr} - \frac{\rho u}{r^2} = 0,$$

where p is the pressure, ρ the density, u the velocity and r the radius. [3]

A6. A student leaves home at 0600 hours to catch a train which leaves at 0630 hours. To get to the station the student first walks to the bus stop and this takes 5 minutes if it is not raining or 6 minutes if it is raining. The probability that it is raining is 0.4. Buses arrive with a mean time of 6 minutes between any two buses. The bus takes an average of 15 minutes to get to the station if it is not raining or 20 minutes if is raining. It is, therefore assumed that both the time between the arrival of the buses at the bus stop and the arrival time of the bus follows the exponential model.

(a) Write down at least two more assumptions of the model. [1]

(b) Use as many of the random numbers given below to find out whether the student catches the train.

0.532 0.325 0.658 0.729 0.091 [4]

SECTION B: Answer TWO questions in this section [60].

B7. The logistic growth equation for the natural growth rate of a population is

$$\frac{dN}{dt} = rN(K - N),$$

where $N(t)$ is the size of the population at time t and K is the carrying capacity of the population.

(a) With the aid of the graph of $\frac{dN}{dt}$, show that the population converge to the carrying capacity, K, as $t \to \infty$. [3]

The above population is then subjected to harvesting by an amount denoted by $h(N)$. The model is then modelled by

$$\frac{dN}{dt} = rN(K - N) - EN$$

where $f(N) = rN(K - N)$ and $h(N) = EN$ are the natural growth rate of the population and the rate of reduction of the population respectively.

(b) Find the equilibrium points of

$$\frac{dN}{dt} = rN(K - N) - EN.$$

Hence sketch the graph of

$$\frac{dN}{dt} = rN(K - N) - EN$$

as a function of N for cases $E < rK$ and $E > rK$ showing the nature of the equilibrium point. [7]

(c) Sketch the graphs of of the equilibrium point as a function of E, showing on the graphs the nature of the population when $E < rK$ and $E > rK$.

What happens to the population if E is increased above rK? [5]

(d) Write down the bifurcation point. [2]

(e) Show that a strictly positive sustainable yield is only possible if $E < rK$. [3]

(f) Write down the value of E that maximizes the maximum sustainable yield. Hence write down the maximum sustainable yield. [4]

(g) Show that when the maximum sustainable yield is maximum the stable population is half the carrying capacity. [3]

(h) Sketch a graph of the maximum sustainable yield as a function of E showing the maximum sustainable yield and where it occurs. [3]

B8. The water supply for Bulawayo is provided by a series of dams situated at various points around the city. The water flows, under the force of gravity, along pipelines from the dams to a central reservoir on the edge of the town.

The town council would like an estimate of how the depth of water in the dams is likely to change throughout the dry season, during which time no significant rainfall is expected. There is very little change in altitude between the start of any pipeline in the dam wall and its finish at the cental reservoir.

The dams are sited in valleys with steep sides and have a rectangular shape. All the sides of the dams and the dam walls can be considered vertical.

(a) List, at least, six factors which you think may be relevant in a model for the depth of a fluid in such a dam. [5]

(b) Write down the dimensions of each of the factors identified above and categorize the factors as constant, parameter, input or output variable. [4]

(c) It is given, from the conservation of water volume, that a simple model for the rate of change of depth of fluid in a dam is

$$\frac{dh}{dt} = -\lambda v,$$

where v is the average velocity of water leaving the pipeline. Assuming that v, is only a function of the depth of water in the dam, the density of the fluid and the acceleration due to gravity, use dimensional analysis to find an expression for v. Hence find an expression for the depth of water in the dam as a function of time. [6]

(d) In one particular dry season, the depth of water in one of the dams as a function of time was as follows By drawing an appropriate graph, use these data to check the validity of the model developed above. [7]

Depth (metres)	20.2	18.9	17.6	16.4	15.2	10.9
Time (months)	0	1	2	3	5	8

(c) An alternative model of such flow suggests that the product of v and the cross-sectional area of the pipeline should be a function of the density of the water, the viscosity of the water, the acceleration due to gravity and the ratio of the depth of water in the dam to the length of the pipeline. Use dimensional analysis to investigate the relationship between v and the other factors in this model. [6]

B9. (a) Given that 0.134, 0.342, 0.521 and 0.854 are random numbers from a uniform distribution on [0,1], generate four random numbers from the discrete distribution of random variable X [4]

Value(x)	0	1	2	3	4	5
P(X = x)	0.1	0.25	0.2	0.3	0.1	0.05

(b) In a certain city the daily consumption of water, X litres is a random variable, but tends to be mainly around $x = 3$. Small and large values of X are very rare. A mathematical modeller decides that a suitable model for the probability density function would be of the form

$$f(x) = \begin{cases} kxe^{-bx}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

(i) What would be suitable values for the parameters k and b? [5]

(ii) Find the cumulative distribution function $F(x)$ and use it to simulate a value X from RND = 0.445. [6]

(c) A firm is carrying out a cost-cutting exercise and requires your help with an investigation into how it can reduce its transport costs. The firm employs a number of drivers who cover a substantial amount of mileage every day. There has recently been a large increase in their fuel costs and drivers can achieve a higher rate of kilometrs per litre from their vehicles by driving at a lower speed.This, however increase journey times and the cost of the drivers' time.Given that the vehicles deliver 40km per litre at 60km/h decreasing steadily with increasing speed to 20 km per litre at 100km/h, construct a model which, given relevant information, could give advice on the optimum driving speed to keep costs to a minimum. [15]

END OF QUESTION PAPER