NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF APPLIED MATHEMATICS

SMA2209/CIN2215 MATHEMATICAL MODELLING

MAY 2005

Time : 3 hours

This paper contains TWO sections. Answer ALL questions in section A and THREE questions from section B.

SECTION A: Answer ALL questions in this section [40].

A1. Write down and explain **five** modelling skills that are required when constructing a mathematical model. [10]

A2. At time $t = 0$ a bullet of mass m is fired vertically into the air with speed U. A model is required to predict the highest point it reaches and the time it takes to reach the highest point. Two simple models based on a uniform vertical acceleration due to gravity can be derived by

1. assuming that force due to air resistance is $k_1 \times$ the velocity,
2. assuming that force due to air resistance is $k_2 \times$ the square of velocity.

Model 1 gives the predictions

$$(i)H = \frac{mU}{k_1} - \frac{(m^2 g)}{k_1^2} \ln\left\{1 + \frac{k_1 U}{(mg)}\right\}$$

$$(ii)T = \frac{m}{k_1} \ln\left\{1 + \frac{k_1 U}{(mg)}\right\},$$

k_1 is a parameter.

Model 2 gives the predictions

$$(i)H = \frac{0.5m}{k_2}\ln\left\{1+\frac{k_2U^2}{(mg)}\right\}$$

$$(ii)T = \sqrt{\frac{m}{k_2 g}}\tan^{-1}\left\{U\sqrt{\frac{k_2}{(mg)}}\right\},$$

k_2 is a parameter.

(a) Show that $[k_1] = MT^{-1}$ and $[k_2] = ML^{-1}$. [3]

(b) Hence check these models for consistency. [7]

A3. In the investigation of a homicide or accident death it is often important to estimate the time of death. From experimental observations it is known that, to an accuracy satisfactory in many circumstances, the surface temperature of an object changes at a rate proportional to the difference between the temperature of the object and that of the surrounding environment (the ambient temperature). This is known as Newton's law of cooling. If $\theta(t)$ is the temperature of the object at time t, and T is the constant ambient temperature, show that θ satisfy the linear differential equation

$$\frac{d\theta}{dt} = -k(\theta - T),$$

where $k > 0$ is a constant of proportionality. [2]

Suppose that at time $t = 0$ a corpse is discovered, and that its temperature is measured to be θ_o. Assuming that at the time of death the body temperature had the normal value of 98.6^oF and that the above differential equation is valid, determine the time of death, t_d. given that the temperature of the corpse is 85^o when discovered and 74^oF two hours later, and that the ambient temperature is 65^oF. [6]

A4. Each batch of a chemical used in drug manufacture is tested for impurities. The percentage of impurity is X. A mathematical model decides that a suitable model for the probability density function would be given by

$$f(x) = \begin{cases} kx, & 0 < x \leq 1 \\ \frac{1}{3}k(4-x), & 1 < x \leq 4 \\ 0 & \text{otherwise.} \end{cases}$$

where k is a constant.

(a) Find a suitable value of k. [2]

(b) Find the cumulative distribution function, $F(x)$. [5]

(c) Simulate the percentage of impurity if

(i) $RND = 0.2$,

(ii) $RND = 0.5$. [5]

SECTION B: Answer B5 and any TWO questions in this section [60].

B5. Answer **one** of the following questions.

(a) A firm is carrying out a cost-cutting exercise and requires your help with an investigation into how it can reduce its transport costs. The firm employs a number of drivers who cover a substantial amount of mileage every day. There has recently been a large increase in their fuel costs and drivers can achieve a higher rate of kilometers per litre from their vehicles by driving at a lower speed.This, however increase journey times and the cost of the drivers' time.Given that the vehicles deliver 40km per litre at 60km/h decreasing steadily with increasing speed to 20 km per litre at 100km/h, construct a mathematical model which, given relevant information, could give advice on the optimum driving speed to keep costs to a minimum. [20]

(b) As a result of adverse claims experience, considered to be a random fluctuation, a company finds that its free resources (capital) have fallen below the level indicated by the desired ruin probability. Consider the effect on premium rates etc. of the the corrective measures:

(i) additional capital,

(ii) adjustment of the reinsurance excess level,

(iii) merger with another similar company.

What considerations arise if the adverse experience is a permanent feature rather than a fluctuation?

(i) Taking an example of a company of your choice, how would you tackle this task?

(ii) How might mathematical models help in this work?

(iii) Formulate a mathematical model that you can use. [20]

(c) What are the main purposes of business planning model? How would you use such a model to investigate the relative importance of the various data required by a business in its planning function.

(i) Taking an example of a company of your choice, how would you tackle this task?

(ii) How might mathematical models help in this work?

(iii) Formulate mathematical model that you can use. [20]

B6. A model for growth rate of a population with depensation is given by

$$g(N) = r(N - K_c)(K - N),$$

where N denote the size of the population , r and K are positive constants, K_c is a positive or negative constant which is less than K. The differential equation is therefore

$$\frac{dN}{dt} = f(N) = rN(N - K_c)(K - N).$$

(a) Find the equilibrium points for $f(N)$. [2]

(b) Sketch the graphs of $g(N)$ and $f(N)$ for cases $K_c < -K$; $-K < K_c < 0$ and $0 < K_c < K$. [6]

(c) On the graphs of $f(N)$ indicate the equilibrium points and their nature. [2]

(d) Sketch the solution curves of $f(N)$. [2]

(e) When do we have depensation and when is critical? [2]

(f) Compute the recovery time T_R of the equilibrium $N = K$ with $K_c < K$, [3]

(g) Sketch the graph of T_R as a function of K_c, holding all other parameters constant. [2]

(h) What happens to T_R as K_c approaches K?. [1]

B7. A reservoir is needed to provide a constant supply of $60000m^3$ of water every day. The monthly inflow rate (10^6) is variable, but the following data are available:

Month)	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Mean	1.5	1.8	2.1	1.2	1.0	0.6	1.5	2.1	2.4	1.5	1.8	2.4
SD	0.5	0.6	0.8	1.0	0.8	0.4	0.5	0.6	0.8	1.0	0.6	0.5

It is assumed that each month's inflow is a random variable with with a normal distribution. The mean μ and standard deviation(SD) σ are given in the data. Using the following random numbers:

0.577	0.976	0.525	0.128	0.887	0.123
0.649	0.229	0.112	0.375	0.683	0.006
0.431	0.318	0.285	0.236	0.865	0.041
0.108	0.763	0.537	0.849	0.448	0.311
0.442	0.565	0.917	0.799	0.266	0.271
0.537	0.356	0.536	0.006	0.573	0.108

(a) simulate the monthly flows (10^6) for the next 12 months, [8]

(b) calculate the net flows into the reservoirs for each month, [5]

(c) the cumulative flows for the year, [3]

(d) calculate the least capacity needed to avoid running dry. [4]

B8. The following table gives the estimated number of fish y in a lake over a number of years.

Time(years)	0	1	2	3	4	5	6
$y(\times 10^3)$	500	700	820	890	930	960	980

(a) Show that y can be reasonably represented by a logistic model

$$\frac{L}{y} - 1 = \left(\frac{L}{y_0} - 1\right)e^{-kt},$$

and obtain estimates for the parameters L and k. [12]

(b) Compare the model's prediction with the actual data. [8]

END OF QUESTION PAPER