

FACULTY OF APPLIED SCIENCES
DEPARTMENT OF APPLIED MATHEMATICS
SMA 4101: ADVANCED LINEAR ALGEBRA

AUGUST 2004 SUPPLEMENTARY EXAMINATIONS

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY THREE** questions from Section B.

SECTION A: [25 marks]

Answer **ALL** questions from this section.

- A1.** (a) Define a linear map. [1]
(b) Let V be the vector space of n -square matrices over \mathbb{R} and let M be a fixed matrix in V . Define the mapping $T : V \rightarrow V$ by $T(A) = AM + MA$ where $A \in V$. Show that T is linear. [3]
(c) Let $F : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be the linear mapping defined by

$$F(x, y, s, t) = (x - y + s + t, x + 2s - t, x + y + 3s - 3t).$$

Find a basis for and the dimension of $\ker F$. [6]

- A2.** Suppose that U and V are finite dimensional vector spaces and $F : V \rightarrow U$ is a linear map. Prove that $\dim V = \dim(\text{Im } F) + \dim(\ker F)$. [8]

- A3.** Let T be the operator on \mathbb{R}^3 defined by $T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$.

- (a) Show that T is invertible. [4]
(b) Find a formula for T^{-1} . [3]

- B4. (a) Prove that the sequence $x_n = \frac{(-1)^n n + 10}{\sqrt{n^2 + 1}}$, ($n \in \mathbb{N}$); is bounded. [6]
- (b) Define a monotonic sequence. [3]
- (c) Show that the sequence $x_n = \left(1 + \frac{1}{n}\right)^n$ is strictly increasing. [7]
- (d) Show that every unbounded sequence contains a subsequence which is strictly increasing. [9]
- B5. (a) Let $S(p)$ denote the sum of the digits of the natural number p . (If $p \leq 9$ then $S(p) = p$).
- (i) Let $x_1 \in \mathbb{N}$; $x_{n+1} = S(x_n)$ $n \in \mathbb{N}$.
Show that there exists some $N \in \mathbb{N}$ such that $\forall m, n \geq N$, then $x_n = x_m$. [6]
- (ii) Let $x_1 \in \mathbb{N}$; $x_{n+1} = S(S(x_n) + x_n)$; $n \in \mathbb{N}$
Calculate x_{1984} if $x_1 = 1983$. [5]
- (b) Prove that $\lim_{n \rightarrow \infty} x_n = a$ if and only if $\forall n \in \mathbb{N}$, $x_n = a + \alpha_n$ where $\{\alpha_n\}$ is a sequence which is infinitely small. [6]
- (c) Prove that $\lim_{n \rightarrow \infty} x_n = 1$, where $x_n = \frac{n}{n+1}$. [3]
- (d) Suppose that $\lim_{n \rightarrow \infty} x_n = 0$ and $x_n > -1 \forall n \in \mathbb{N}$.
If $p \in \mathbb{N}$, prove that $\lim_{n \rightarrow \infty} \sqrt[p]{1 + x_n} = 1$. [5]
- B6. (a) Define the upper limit and lower limit of a sequence $\{X_n\}$. [4]
- (b) State the Bolzano-Weierstrass theorem. [2]
- (c) Use the Weierstrass theorem to prove that:
- (i) the sequence $\{x_n\}$, where $x_1 = 0$; $x_{n+1} = \sqrt{6 + x_n}$; ($n \in \mathbb{N}$); has a finite limit.
Calculate this limit. [5]
- (ii) the sequence $\{x_n\}$ where $x_n = \left(1 + \frac{k}{n}\right)^n$ has a finite limit.
Calculate this limit. [4]
- Hint: Use the fact that $\lim_{p \rightarrow \infty} \left(1 + \frac{1}{p}\right)^p = e$. [2]
- (d) Calculate the upper and lower limits of $x_n = \frac{n^2 \sin\left(\frac{n\pi}{2}\right) + 1}{n+1}$. [5]
- B7. (a) State the Cauchy condition for convergence of a sequence. [2]
- (b) Prove that the sequence $x_n = \sum_{k=1}^n \frac{1}{k}$ ($n \in \mathbb{N}$) diverges. [5]
- (c) Calculate the following limits:

$$(i) \lim_{n \rightarrow \infty} \left(\frac{2^n + 1}{2^n} \right)^{2^n} \quad [3]$$

$$(ii) \lim_{n \rightarrow \infty} \left(\frac{2002}{n} \right)^n \quad [3]$$

$$(iii) \lim_{n \rightarrow \infty} \sqrt{n^2 + n} - \sqrt{n^2 - n}. \quad [3]$$

$$(iv) \lim_{n \rightarrow \infty} \frac{10^n + n!}{2^n + (n+1)!} \quad [3]$$

(d) Use the comparison test to determine whether $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ converges or diverges. [6]

B8. (a) State the following convergence tests for series:

(i) D'Alembert's criterion [2]

(ii) Cauchy's integral test [2]

(iii) Leibniz's convergence test. [2]

(b) Investigate the convergence of:

$$(i) \sum_{n=1}^{\infty} \frac{1}{n!} \quad [3]$$

$$(ii) \sum_{n=1}^{\infty} \left(\frac{3n}{3n+1} \right)^n \quad [3]$$

$$(iii) \sum_{n=1}^{\infty} \frac{3^n n!}{n^n} \quad [3]$$

$$(iv) \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2(n+2)^2} \quad [3]$$

(c) Prove that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{n}$ converges uniformly in all its convergence interval $] -1, 1[$ [4]

(d) Expand $f(x) = \cosh x$ in power series of x . Find the interval of convergence of the series and use the Lagrange theorem to find its remainder. [3]

END OF QUESTION PAPER