NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF APPLIED MATHEMATICS

SMA4102 MODERN ALGEBRA

January 2003
Time: 3 Hours

Candidates should attempt ALL questions from Section A and ANY FOUR questions from Section B.

SECTION A

A1. a) When is a subgroup N of a group G said to be a normal subgroup?
b) Let G be a group. Show that

$$Z(G) = \{a \in G : ax = xa \text{ for all } x \in G\}.$$

is a normal subgroup of G.

[3,5]

A2. For each of the following functions, determine whether f is a ring homomorphism. If it is then find its kernel.

a) $f : \mathbf{C} \to \mathbf{R}$, $f(a+ib) = b$, for all $a, b \in \mathbf{R}$

b) $f : \mathbf{Z} \to M_2(\mathbf{Z})$, $f(k) = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ for all $k \in \mathbf{Z}$.

[3,4]

A3. Let n, a, b, c, d, k be integers such that $a \equiv b(\text{mod}\, n)$ and $c \equiv d(\text{mod}\, n)$. Prove that

(i) $(a-c) \equiv (b-d)(\text{mod}\, n)$
(ii) $ka \equiv kb(\text{mod}\, n)$

[6]

A4. Calculate the *gcd* of $x^3 + 2x^2 + 4x - 7$ and $x^2 + x - 2$ in $\mathbf{Q}[x]$.

[5]

A5. Let a, b and n be integer. If $(a,b) = d$, show that $(an, bn) = dn$.

[8]

A6. Let G be a group. Show that the function $f : G \to G$ defined by $f(x) = x^{-1}$ is an automorphism of G if and only if G is abelian.

[6]

SECTION B

B7. a) Define $\phi : M_2(\mathbf{Z}) \to \mathbf{Z}$ and $\psi : \mathbf{Z} \to M_2(\mathbf{Z})$ by

$$\phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \text{ and } \psi(z) = \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix}.$$

Determine which of ϕ, ψ (if either) is a homomorphism.

b) Let $\phi : R \to S$ be a homomorphism. Prove that $\ker \phi$ is an ideal of R and $\phi(R) = \{\phi(r) : r \in R\}$ is a subring of S. Give a specific example to show that, even if R, S have multiplicative identities 1_R and 1_s we may have $\phi(1_R) \neq 1_s$.

[15]

B8. Let $\mathbf{Z}[i]$ denote the set $\{a + ib : a, b \in \mathbf{Z}\}$ of complex numbers.

a) Defining $N : \mathbf{Z}[i] \to \mathbf{Z}$ by $N(a+ib) = a^2 + b^2$, prove that for $\alpha, \beta \in \mathbf{Z}[i]$, $N(\alpha\beta) = N(\alpha)N(\beta)$.
Deduce that if $m, n \in Z$ each is a sum of two squares then so is mn. Write 37, 89 and 3293 as sums of two squares..

b) Prove that, given $\alpha, \beta (\neq 0)$ in $\mathbf{Z}[i]$, there exists $m, r \in \mathbf{Z}[i]$ such that $\alpha = m\beta + r$ $0 \leq N(r) < N(\beta)$. Deduce that, in the ring $\langle \mathbf{Z}[i], +, \bullet \rangle$ each ideal is a principal ideal.

c) Assuming that each prime $p \in \mathbf{Z}$ which is of the form $4k + 1 \in \mathbf{Z}$, $k > 0$ can be expressed as a sum $p = l^2 + m^2$ of two squares, show briefly that, if $p = u^2 + v^2$ then either (i) $u = \pm l$ and $v = \pm M$ (ii) $u = \pm \mathrm{M}$ and $v = \pm l$.

[Any facts you use concerning $\langle \mathbf{Z}[i], +, \bullet \rangle$ should be clearly stated.]

B9. Let $N : \mathbf{Z}[\sqrt{-5}] \to \mathbf{Z}$ defined by

$$N(m + n\sqrt{-5}) = m^2 + 5n^2$$

Show that if $\alpha, \beta \in \mathbf{Z}[\sqrt{-5}]$,

$$N(\alpha\beta) = N(\alpha)N(\beta)$$

Deduce that u is a unit in $\mathbf{Z}[\sqrt{-5}]$ if and only if $N(u) = \pm 1$.

By considering $(1 + \sqrt{5})(1 - \sqrt{-5})$ show that 3 is irreducible in $\mathbf{Z}[\sqrt{-5}]$ but not prime.

[15]

B10. a) Prove directly that $D = \mathbf{Z}[\sqrt{-1}]$ is a Eulidean ring. (You may assume that D is an integral domain).

b) Find a greatest common divisor d of $a = 11 - 10i$, $b = 3 + 5i$, and find s, t in D such that $d = as + bt$.

c) Express 5 as a product of prime elements of D. Verify that the factors in this product really are prime elements.

[15]

B11. a) Explain what is meant by a homomorphism f of a ring R to a ring S, and define the kernel of f. Show that $f(0) = 0$, and $f(-a) = -f(a)$ all $a \in R$.

b) If g is a homomorphism of R onto S, and $N = \ker g$, prove that N is an ideal of R, and that $R/N \approx S$.

c) Let A and B be ideals of R, and define

$$A + B = \{a + b : a \in A,\ b \in B\}.$$

Assuming that $A + B$ is a ring, A is an ideal of $A + B$, and $A \cap B$ is an ideal of B, write down a typical element of the quotient ring $(A+B)/A$. Hence find a homomorphism $\lambda : B \to (A+B)/A$, and deduce that $(A+B)/A \approx B/(A \cap B)$.

[15]

B12. (i) Define the centre of a group and show that it is a normal subgroup of the group.

(ii) Let G be a finite group. Show that if G is cyclic, then G is abelian.

(iii) Let G be a group with a subgroup of $H = \{e, h\}$. Show that if H is a normal subgroup of G then h belongs to the centre of G.

(iv) Define the centralizer $C(x)$ of an element x in a group and show that it is a subgroup of the group.

(v) Show that $C(yxy^{-1}) = yC(x)y^{-1}$.

[15]

END OF QUESTION PAPER.