

DEPARTMENT OF APPLIED MATHEMATICS

SMA 4102: MODERN ALGEBRA

APRIL/MAY 2003

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY THREE** questions from Section B.

SECTION A [25 marks]Answer **ALL** questions from this section.

- A1. Let m be a natural number greater than 1 and let $\epsilon = \cos \frac{2\pi}{m} + i \sin \frac{2\pi}{m}$. Define the set $P = \{\epsilon^n : n \in \mathbb{N}\}$.
Prove that P with the usual multiplication of complex numbers is a cyclic group. [9]
- A2. Let m be a positive integer. Prove that congruence modulo m is compatible with addition and subtraction of natural numbers, that is, for natural numbers n_1, n'_1, n_2, n'_2 , if $n_1 \equiv n'_1 \pmod{m}$ and $n_2 \equiv n'_2 \pmod{m}$ then
- (a) $n_1 + n_2 \equiv n'_1 + n'_2 \pmod{m}$, [4]
(b) $n_1 \cdot n_2 \equiv n'_1 \cdot n'_2 \pmod{m}$. [4]
- A3. Assume that G is a group, with identity e and $a \in G$ and that there exist unequal integers r and s such that $a^r = a^s$. Prove that there is a smallest positive integer n such that $a^n = e$, the neutral element of G . [8]

SECTION B [75 marks]

Answer ANY THREE questions from this section. Each question carries 25 marks.

- B4.** (a) Define an isomorphism from a group (G, \cdot) onto a group $(H, \#)$. [3]
 (b) Prove that isomorphism of groups is an equivalence relation in the class of all groups. [9]
 (c) Let G and H be two groups and $f : G \rightarrow H$ be an isomorphism. Prove that
 (i) if e is the neutral element of G , then $f(e) = \hat{e}$ is the neutral element of H , [4]
 (ii) if a^{-1} is the inverse of an element a of G then $[f(a)]^{-1}$ is the inverse of $f(a)$ in H , [4]
 (iii) if G is Abelian then H is Abelian. [5]
- B5.** (a) Let $\mathbb{Z}[\sqrt{-3}] = \{a + b\sqrt{-3} : a, b \in \mathbb{Z}\}$.
 Prove that $\mathbb{Z}[\sqrt{-3}]$ with addition and multiplication is an integral domain. [15]
 (b) Define a mapping f from the ring of integers to the ring of matrices with integer coefficients as

$$f : a \rightarrow A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$
 Show that f is a ring isomorphism. [10]
- B6.** (a) Define an equivalence relation on the set of integers by letting $a \sim b$ mean that either both a and b are negative or both a and b are nonnegative. There are two equivalence classes: $[-1]$ consisting of the negative integers and $[0]$ consisting of the nonnegative integers. Define $[+]$ on the set $A = \{[-1], [0]\}$ of equivalence classes by $[a] [+][b] = [a + b]$. Show that $[+]$ is not an operation on A . [6]
 (b) Prove that a group G is Abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$. [9]
 (c) Let R be a ring and $a, b, c \in R$, show that each of the equations $a + x = b$ and $x + a = b$ has a unique solution. [4]
 (d) Assume that $\alpha : S \rightarrow T$ and $\beta : T \rightarrow U$ are mappings. Prove that if α and β are onto, then $\beta \circ \alpha$ is onto. [6]

- B7. (a) Let $F = M(\mathbb{R})$ denote the set of all functions (mappings) $f : \mathbb{R} \rightarrow \mathbb{R}$. For each $x \in \mathbb{R}$, $f, g \in F$ define the operations $f + g$ and $f \cdot g$ by
 $(f + g)(x) = f(x) + g(x)$ and $(f \cdot g)(x) = f(x) \cdot g(x)$
 Verify that F with the operations $+$ and \cdot as defined is a ring. [12]
- (b) Let D be an integral domain. Show that if $a, b, c \in D$, $a \neq 0$ and $ab = ac$ then $b = c$. [3]
- (c) Prove that if D is an integral domain of characteristic 0, then D contains a subring isomorphic to \mathbb{Z} . [10]
- B8. (a) Show that there is no rational number x such that $x^2 = 2$. [6]
- (b) Let z^* denote the conjugate of a complex number z . Let Q denote the set of matrices in $M(2, \mathbb{C})$ that have the form $\begin{pmatrix} z & w \\ -w^* & z^* \end{pmatrix}$. For example
 $\begin{pmatrix} 1 + 2i & 2 + i \\ -2 + i & 1 - 2i \end{pmatrix}$ is in Q . Prove that Q is a division ring, that is a ring with a unity in which each non-zero element has an inverse relative to multiplication. [11]
- (c) Let (G, \cdot) and $(H, \#)$ be two groups and $\theta : G \rightarrow H$ be a homomorphism. Prove that $\text{Ker}\theta$ is a normal subgroup of G . [8]

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END OF QUESTION PAPER