NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY
SMA 4102

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

SMA 4102: MODERN ALGEBRA

JUNE 2004

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY THREE** questions from Section B.

SECTION A: [25 marks]

Answer **ALL** questions from this section.

A1. Prove that the permutation group (S_3, o), is not Abelian . [9]

A2. Let m be a positive integer. Prove that congruence modulo m is compatible with addition and subtraction of natural numbers, that is, for natural numbers n_1, n_1', n_2, n_2', if $n_1 \equiv n_1'(\text{mod } m)$ and $n_2 \equiv n_2'(\text{mod } m)$ then

(a) $n_1 + n_2 \equiv n_1' + n_2'(\text{mod } m)$, [4]

(b) $n_1.n_2 \equiv n_1'.n_2'(\text{mod } m)$. [4]

A3. Assume that G is a group, with identity e and $a \in G$ and that there exist unequal integers r and s such that $a^r = a^s$. Prove that there is a smallest positive integer n such that $a^n = e$, the neutral element of G. [8]

SECTION B: [75 marks]

Answer **ANY THREE** questions from this section. Each question carries 25 marks.

B4. (a) Define an isomorphism from a group $(G, .)$ onto a group $(H, \#)$. [3]

(b) Prove that isomorphism of groups is an equivalence relation in the class of all groups. [9]

(c) Let G and H be two groups and $f : G \to H$ be an isomorphism. Prove that

(i) if e is the neutral element of G, then $f(e) = \tilde{e}$ is the neutral element of H, [4]

(ii) if a^{-1} is the inverse of an element a of G then $[f(a)]^{-1}$ is the inverse of $f(a)$ in H, [4]

(iii) if G is Abelian then H is Abelian . [5]

B5. (a) Let $\mathbb{Z}[\sqrt{-3}] = \{a + b\sqrt{-3} : a, b \in \mathbb{Z}\}$.
Prove that $\mathbb{Z}[\sqrt{-3}]$ with addition and multiplication is an integeral domain. [15]

(b) Define a mapping f from the ring of integers to the ring of matrices with integer coefficients as

$$f : a \to A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

Show that f is a ring isomorphism.[*Consider the usual operations of addition and multiplication*] [10]

B6. (a) Define an equivalence relation on the set of integers by letting $a \sim b$ mean that either both a and b are negative or both a and b are nonnegative. There are two equivalence classes: $[-1]$ consisting of the negative integers and $[0]$ consisting of the nonnegative integers. Define $[+]$ on the set $A = \{[-1], [0]\}$ of equivalence classes by $[a]\ [+]\ [b] = [a + b]$. Show that $[+]$ is not an operation on A. [6]

(b) Prove that a group $(G, .)$ is Abelian if and only if $(ab)^{-1} = b^{-1}a^{-1}$ for all $a, b \in G$. [9]

(c) Let $R(, +, .)$ be a ring and $a, b, c \in R$, show that each of the equations $a + x = b$ and $x + a = b$ has a unique solution. [4]

(d) Assume that $\alpha : S \to T$ and $\beta : T \to U$ are mappings. Prove that if α and β are onto, then $\beta \circ \alpha$ is onto. [6]

B7. (a) Let $F = M(\mathbb{R})$ denote the set of all functions(mappings) $f : \mathbb{R} \to \mathbb{R}$. For each $x \in \mathbb{R}$, $f, g \in F$ define the operations $f + g$ and $f.g$ by
$(f+g)(x) = f(x) + g(x)$ and $(f.g)(x) = f(x).g(x)$
Verify that F with the operations $+$ and $.$ as defined is a ring. [12]

(b) Let D be an integral domain. Show that if $a, b, c \in D$, $a \neq 0$ and $ab = ac$ then $b = c$. [3]

(c) Let E be a given non-empty set and let A and B be subsets of E. Prove that $(P(E), o)$ is an Abelian group, where $P(E)$ is the power set and $'o'$ is an operation defined by $AoB = (A \cap B^c) \cup (B \cap A^c)$. [10]

B8. (a) Show that there is no rational number x such that $x^2 = 2$. [6]

(b) Let z^* denote the conjugate of a complex number z. Let Q denote the set of matrices in $M(2, \mathbb{C})$ that have the form $\begin{pmatrix} z & w \\ -w^* & z^* \end{pmatrix}$. For example
$\begin{pmatrix} 1+2i & 2+i \\ 2-i & 1-2i \end{pmatrix}$ is in Q. Prove that $(Q, +, .)$ is a division ring, that is a ring with a unity in which each non-zero element has an inverse relative to multiplication. [11]

(c) If $\sim$ is an equivalence relation on a set S, prove that the set of equivalence classes of $\sim$ forms a partition of S. [8]

END OF QUESTION PAPER