NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 4103

DEPARTMENT OF APPLIED MATHEMATICS

SMA 4103 INTRODUCTION TO FLUID DYNAMICS

MAY 2005 EXAMINATIONS

Time : 3 hours

Candidates must attempt ALL questions in section A and any TWO questions in section B.

SECTION A: Answer ALL questions in this section [50].

1. Consider the two-dimensional flow

$$\vec{V} = [u, v, w] = [\frac{1}{(1+t)}, 1, 0], \quad \text{for} \quad t > -1.$$

Find and sketch

(a) the streamline at $t = 0$ which passes through the point $(1, 1, 0)$; [3]

(b) the path of a fluid particle which is released from $(1, 1, 0)$ at $t = 0$; [3]

(c) the position at $t = 0$ of a streak of dye released from $(1, 1, 0)$ during the time interval $-1 < t < 0$. [3]

2. A steady two-dimensional flow, is given by $\vec{V} = (\alpha x, -\alpha y, 0)$, with α a constant.

(a) Find the equation for the general streamline of the flow, and sketch a few streamlines. [2]

(b) At $t = 0$, the fluid on the curve $x^2 + y^2 = a^2$ is marked with a dye. Find the equation which governs the shape of the dye fluid for $t > 0$. [4]

(c) Does the area enclosed within the dyed region change over time? Give reasons for your answer. [3]

3. Consider a flow field such that all variables are independent of y and z such that $\vec{V} = (u(x,t), 0, 0) = u(x,t)\hat{\imath}$. Suppose that there are no external forces, and that the pressure $p \to p_0$, $u \to u_0$ as $|x| \to \infty$ where p_0 is constant.

   (a) Show that Euler's equation and the continuity equation reduces to

   $$\rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}) = -\frac{\partial p}{\partial x} \quad \text{and} \quad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0.$$

   [4]

   (b) Use these equations to prove that

   $$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + p) = 0.$$

   [3]

   (c) Deduce that

   $$\frac{d}{dt}\int_{-\infty}^{\infty} \rho u dx = 0.$$

   This last expression is an expression of the conservation of momentum. [4]

4. Given that a fluid flow is described by the velocity field given by

   $$\vec{V} = -\alpha(3y\hat{\imath} - x\hat{\jmath}),$$

   find:

   (a) the vorticity, [2]

   (b) the rate of strain tensor, [2]

   (c) the principal rates. [3]

5. A two-dimensional flow occupies the half-space $y < 0$ and is given by the velocity potential $\phi = e^{ky}\sin kx$.

   (a) Show that the flow is incompressible. [2]

   (b) Calculate the velocity field $\vec{V}$ and the streamfunction $\psi(x,y)$. [3]

   (c) Sketch the streamlines. [3]

6. Let $W(z) = \phi + i\psi$ represent the complex potential for some flow.

   (a) Prove that $\nabla\phi \cdot \nabla\psi = 0$. [3]

   (b) Hence, or otherwise show that the lines of constant ϕ and ψ are perpendicular. [3]

SECTION B: Answer TWO questions in this section [50].

7. Whether a fluid is incompressible or not, each fluid element must conserve its mass as it moves.

   (a) Consider the rate of flow through a fixed surface S, drawn in the fluid, and advance arguments to show that this conservation of mass implies

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V})$$

[8]

   (b) Show too that this equation may alternatively be written as

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{V} = 0$$

[4]

   (c) It follows that$\nabla \cdot \vec{V} = 0$ when $\frac{D\rho}{Dt} = 0$. What does it mean exactly and does it make sense? [3]

   (d) Determine if the flow field represented by the velocity components

$$u = \frac{y-b}{(x-a)^2 + (y-b)^2}, \quad \text{and} \quad v = \frac{y-a}{(x-a)^2 + (y-b)^2}$$

   satisfies the continuity equation in (c) above. For this flow, find the streamfunction, $\psi(x, y)$, such that $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$. [6]

   (e) Hence, sketch the streamlines for the flow given in (d). [4]

8. (a) Consider a fluid moving in a velocity field given by:

$$u = -\beta x - yrf(t), \quad v = -\beta y + xrf(t), \quad w = 2\beta z,$$

where $r^2 = x^2 + y^2$ and β is a constant.

(i) Calculate the vorticity of this flow. [3]

(ii) By taking the curl of Euler's equation for a fluid of constant density, it can be shown that the vorticity equation is given by:

$$\frac{D\vec{\omega}}{Dt} = \vec{\omega} \cdot \nabla \vec{V}$$

show that this vorticity equation is satisfied by the flow with the given velocity field if $f(t) = e^{3\beta t}$ [4]

(iii) Calculate the velocity components of the flow u_r and u_θ in cylindrical coordinates. Now consider a material curve (that is, a curve marked with a dye) in the plane $z = 0$, which is initially a circle of radius $a = a_0$. Show that this curve remains in the plane $z = 0$ and its radius is given by $a(t) = a_0 e^{-\beta t}$. [4]

(iv) Verify the circulation theorem for this material curve. [4]

(b) A sphere of radius a moves with constant velocity U along a straight line in an inviscid fluid of constant density which otherwise at rest

(i) Write the equation of motion of the fluid past the sphere. [4]

(ii) In a frame of reference moving with the centre of the sphere, write down the conditions at infinity and on the sphere for a potential ϕ governing the flow. Hence use seperation of variables in spherical polar coordinates to show that

$$\phi = -U(r + \frac{a^3}{2r^2}) \cos \theta.$$

[6]

9. (a) Consider a fluid in a cylindrical vessel of radius a which is undergoing steady solid-body rotation so that its velocity field is given ny

$$\vec{V} = (-\Omega y, \Omega x, 0).$$

(i) By considering each of the vector components of Euler's equation in the form

$$\rho\frac{D\vec{V}}{Dt} = -\nabla p - \rho g\hat{k}$$

show that to within an additive constant, the pressure distribution is given by

$$p(x, y, z) = \frac{1}{2}\rho\Omega(x^2 + y^2) - \rho g z + C,$$

where C is a constant. [5]

(ii) Hence find the shape of the free surface of fluid (that is, the interface between the rotating fluid and the atmosphere) and show that the fluid at the centre of the vessel is lower than that at the edges by $\frac{\Omega^2 a^2}{2g}$. [4]

(iii) Estimate the difference in heights between the centre and edges of a coffee cup of radius 5cm which is rapidly stirred at a rate of 2 revolutions per second. [3]

(b) Euler's equation for the motion of a fluid can be written in the form:

$$\rho\frac{D\vec{V}}{Dt} = \rho\vec{F} - \nabla P.$$

If the body forces are solely due to gravity, so that the scalar potential $\phi = gz$ exits, show that Bernoulli's equations can be written in the form:

$$\frac{P}{\rho} + \frac{\vec{V}^2}{2} + gz = \text{constant}$$

[6]

(c) Show also that for a rotating frame of reference, Euler's equation can be written as:

$$\frac{\partial\vec{V}}{\partial t} + (\vec{\omega} + 2\vec{\Omega}) \times \vec{V} = \vec{F} - \nabla[\frac{\vec{V}^2}{2} + \frac{P}{\rho} - \frac{1}{2}(\vec{\Omega}^2\vec{r}^2)]$$

[7]

10. Let $W(Z)$ represent the complex potential for some flow.

(a) If the flow has velocity $\vec{V} = (u, v)$, prove or show that $\frac{dW}{dz} = u - iv$. [3]

(b) How is the complex potential $W(z)$ related to the flow and the angle θ made by the streamlines locally. [3]

(c) By deducing an appropriate complex potential, show that the streamlines for an irrotaional flow due to a line source and a line sink of equal strengths, m placed at any two positions $(\pm a, 0)$ in the complex (x, y) plane in a homogeneous, incompressible fluid are arcs. [6]

(d) Using the Milne-Thompson circle theorem find the complex potential for the following flows:

(i) uniform flow past a stationary cylinder, [3]

(ii) cylinder $\mid z \mid= a$ in ambient unifrom stream at angle α to $\vec{ox}$. [3]

(e) Using the theorem of Blasius which enables one to evaluate the resultant force and couple exerted on a unit length of the cylinder, by the moving fluid, show that the force exerted on the infinite cylinder $\mid z \mid= a$ in the irrotational flow is given by $[X, Y]$ per unit length, where $X = 0$, $Y = 2\pi\rho U k$. [7]

11. In unidirectional flow of viscous incompressible fluid the velocity field $u(y, t)$ is found (from the definition of viscousity) to satisfy:

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial P}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2}, \quad where \quad P = P(x, t)$$

(a) Show that if the flow is steady, then

(i) $\partial P/\partial x = dP/dx$ is constant, and [2]

(ii) $\partial^2 u/\partial y^2 = d^2u/dy^2$ is also constant. [2]

(b) If the flow is steady and $dP/dx = -G$, a constant, show that the flow velocity in the channel with rigid impermeable walls $y = 0,\ h$ is given by

$$u = \frac{Gy}{2\mu}(h - y).$$

[7]

(c) Sketch $u(y)$. Is the flow in the right direction? [4]

(d) The plane $y = 0$ is made to oscillate in its own plane so that $u = U\cos(nt)$, where U and n are constants. The fluid occupies the region $y > 0$ and the pressure is uniform. Show that by writing $u = Re[\exp^{int} f(y)]$ the velocity of the fluid is:

$$u = Ue^{-(n/2\nu)^{1/2}y}\cos[nt - (n/2\nu)^{1/2}y]$$

where $\nu = \mu/\rho$. [It may be assumed the flow velocity is bounded by $y \to \infty$]. [10]

END OF QUESTION PAPER