NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

SMA 4107: TIME SERIES ANALYSIS AND SIMULATION

DECEMBER 2004

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY THREE** questions from Section B.

SECTION A: Answer ALL questions in this section [40].

A1. Give a method for simulating a random variable X with cumulative distribution function

$$F(x) = \begin{cases} \frac{1-e^{-2x}+2x}{3}, & 0 < x < 1 \\ \frac{3-e^{-2x}}{3}, & 1 < x < \infty. \end{cases}$$

[6]

A2. (a) Show how to simulate a random variable with the binomial distribution with parameters $n = 6$, $p = 0.4$ using $U \sim U(0,1)$ numbers. [5]

(b) Use the following $U \sim U(0,1)$ numbers to generate values of X; 0.34, 0.89. [2]

A3. Obtain the inverse transform method for the Cauchy distribution (median μ and scale σ) with probability density function given by

$$f(x) = \frac{\sigma}{\pi\{\sigma^2 + (x-\mu)^2\}}.$$

[5]

A4. Show how to simulate a random variable X which has a uniform distribution on the interval (a, b). [5]

A5. Given that $Z_t \sim N(0, \sigma_z^2)$, express the model $X_t = 0.5X_{t-1} + Z_t - 1.3Z_{t-1} + 0.4Z_{t-2}$ in the B notation and determine whether the model is stationary and/or invertible. [2,4]

A6. (a) Find the a.c.f of the second-order MA process given by

$$X_t = Z_t + 0.7Z_{t-1} - 0.2Z_{t-2}$$

where $Z \sim N(0, \sigma_z^2)$. [3]

(b) (i) Find the a.c.f ($\rho(k)$)of the first order AR process given by $X_t - \mu = 0.7(X_{t-1} - \mu) + Z_t$. [4]

(ii) Plot $\rho(k)$, for $k = -6, -5, \ldots, 5, 6$. [4]

SECTION B: Answer THREE questions in this section [60].

B7. (a) Show how the Box-Muller method generates two independent $N(0, 1)$ random variables from two $U(0, 1)$ random variables. [16]

(b) What are the advantages of the Polar-Marsaglia method over the Box Muller method? [4]

B8. Suppose an $AR(2)$ process is given by

$$X_t = \frac{1}{12}X_{t-1} + \frac{1}{12}X_{t-2} + Z_t$$

where $Z_t \sim N(0, \sigma_z^2)$.

(a) Show that X_t is a stationary process. [5]

(b) Show the the ACF of X_t is given by

$$\rho_k = \frac{45}{77}\left(\frac{1}{3}\right)^{|k|} + \frac{32}{77}\left(\frac{-1}{4}\right)^{|k|}, \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots$$

[15]

B9. Given that $Z_t \sim N(0, \sigma_z^2)$;

(a) Show that the AR(2) process

$$X_t = X_{t-1} + cX_{t-2} + Z_t$$

is stationary provided $-1 < c < 0$. Find the autocorrelation function when $c = -3/16$. [10]

(b) Show that the AR(3) process

$$X_t = X_{t-1} + cX_{t-2} + cX_{t-3} + Z_t$$

is non-stationary for all values of c. [10]

B10. Consider the $AR(1)$ process $X_t - \mu = \alpha(X_{t-1} - \mu) + Z_t$, where $\{Z_t\}$ are purely random, and $-1 < \alpha < 1$.

Show that given N observations $x_1, \ldots, x_N$ the **least squares estimates** that minimise

$$S = \sum_{t=2}^{N} [x_t - \mu - \alpha(x_{t-1} - \mu)]^2$$

are given by

$$\widehat{\mu} = \frac{\bar{x}_{(2)} - \widehat{\alpha}\bar{x}_{(1)}}{1 - \widehat{\alpha}}$$

and

$$\widehat{\alpha} = \frac{\sum_{t=1}^{N-1} (x_t - \widehat{\mu})(x_{t+1} - \widehat{\mu})}{\sum_{t=1}^{N-1} (x_t - \widehat{\mu})^2}$$

where $\bar{x}_{(1)}, \bar{x}_{(2)}$ are the means of the first and the last $(N-1)$ observations respectively. [20]