NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 4112

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

SMA4112: MODERN ALGEBRA

MAY 2005

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY THREE** questions from Section B.

SECTION A: Answer ALL questions in this section [40 marks].

A1. Prove that

(a) the inverse of an invertible mapping is invertible, [8]

(b) every field is an integral domain. [6]

A2. Let G be a finite group. Show that each element of G appears once and only once in any given row of the multiplication table. [5]

A3. Assume H to be a subgroup of the group G. Let x, y be two elements in G. Show that the two left cosets xH and yH are equal if and only if $x^{-1}y$ is an element of H. [8]

A4. Write down the multiplication table of S_3 and find the orders of all its elements. [8]

A5. Give an example of a permutation group, G with subgroups H and K such that $H \cup K$ is not a subgroup of G. [5]

SECTION B: Answer THREE questions in this section [60 marks].

B6. (a) If $\sim$ is an equivalence relation on a set X, prove that the set of equivalence classes of $\sim$ forms a partition of S. [14]

(b) Let $X = \{-1, 0, 1\}$ and define the relation $\sim$ on $\mathbb{Z}^+$ by the rule

$$a \sim b \text{ if and only if } a - b \in X.$$

Show that $\sim$ is not an equivalence relation, and give examples to indicate why other properties are not satisfied. [6]

B7. (a) Let m be a natural number greater than 1. Define the set $P = \left\{ e^{\frac{i2\pi n}{m}} : n \in \mathbb{N} \right\}$. Prove that P with usual multiplication of complex numbers is a cyclic group. [8]

(b) Let $\mathbb{C}$ be the field of complex numbers and let $\theta : \mathbb{C} \to \mathbb{C}$ be defined by

$$\theta(a + ib) = a - ib \text{ for all } a, b \in \mathbb{R}.$$

Show that θ is a ring isomorphism from $\mathbb{C}$ to $\mathbb{C}$. [8]

(c) Let $\alpha : \mathbb{C} \to \text{Mat}(2, \mathbb{R})$ be defined by

$$\alpha(a + bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Show that α is a homomorphism. [4]

B8. (a) Let Mat(n, R) be the set of all $n \times n$ matrices with entries from a ring R where n is a positive integer. If $A \in \text{Mat}(n, R)$ and $i, j = \{1, 2, \cdots, n\}$, let A_{ij} be the entry in the i^{th} row and j^{th} column of A. (Thus $A_{ij} \in R$ for each i, j). If it is given thataddition and multiplication on Mat(n,R) is defined by

$$(A + B)_{ij} = A_{ij} + B_{ij}$$

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}, \text{ respectively,}$$

that is, addition is defined componentwise, and for the product the $(i, j)^{th}$ entry of AB is obtained by multiplying the i^{th} row of A by the j^{th} column of B in the usual way. Prove that Mat(n, R) is a ring with these operations. [10]

(b) Let $\phi : G \to H$ be a homomorphism between the groups G and H. Show that ϕ is injective if and only if ker$\phi = \{e\}$. [6]

(c) Let $R(+, .)$ be a ring and $a, b, c \in R$, show that each of the equations $a + x = b$ and $x + a = b$ has a unique solution. [4]

B9. (a) Prove that isomorphism of groups is an equivalence relation in the class of all groups. [10]

(b) Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be a group homomorphism. Show that gof is a group homomorphism. [5]

(c) If H and K are subgroups of G and $x \in G$, prove that $xH \cap xK = x(H \cap K)$. [5]

B10. (a) Prove that the set $D = \{a + b\sqrt{3} : a, b \in \mathbb{Z} \text{ and } a^2 - 3b^2 = 1\}$ with operation multiplication is a group. [8]

(b) Let $\varphi : G \rightarrow H$ be an isomorphism and G is Abelian. Prove that H is also Abelian. [4]

(c) Let a, b, c be elements of a group G. Find the solutions of $x \in G$ of the equations.

(i) $axa^{-1} = e$;
(ii) $axa^{-1} = a$;
(iii) $axb = c$;
(iv) $ba^{-1}xab^{-1} = ba$.

[8]

END OF QUESTION PAPER