NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF APPLIED MATHEMATICS

APRIL 2003 EXAMINATIONS

SMA4135 DYNAMICAL SYSTEMS

Answer ALL Questions in Section A and THREE Questions in Section B

SECTION A: 48 Marks

1. Define

   (a) an equilibrium point of a differential equation equation $dx/dt = f(x)$, [2]

   (b) a fixed point of a map $x_{n+1} = g(x_n)$, [2]

   (c) Given that f and g are both continuously differentiable, state, without proof, sufficient conditions for the linear stability of the equilibrium points in (a) and the fixed point in (b). [2]

2. Write down the equilibrium points of the differential equation

$$dx/dt = a^2 - x^2, \quad a \neq 0, \quad t \geq 0$$

and investigate their stability. [4]

3. Consider the one-dimensional flow

$$\dot{x} = \mu x + \alpha x^3$$

   (a) Locate the equilibrium solutions of this flow. [4]

   (b) For the trivial fixed point $(0, 0)$ sketch the bifurcation diagrams for $\alpha = -1$ and $\alpha = 1$ when μ is the control parameter. [8]

4. Consider the Henon map given by

$$x_{n+1} = 1 + y_n - \alpha x_n^2, \qquad y_{n+1} = \beta x_n$$

   (a) Find the fixed points of this map. [4]

   (b) Discuss the stability of the fixed points when $\alpha = 0.08$ and $\beta = 0.3$ [9]

5. Consider the system of differential equations

$$dx/dt = y, \qquad dy/dt = -x + x^3 - 2\mu y$$

   (a) Find the equilibrium points of this flow. [4]

   (b) Discuss the stability of the equilibrium points [9]

SECTION B: 52 Marks

6. Consider the Lotka-Voltera equations

$$\dot{x} = x[1 - x + ay], \quad \dot{y} = ry[1 + bx - y]$$

   (a) Sketch the phase portraits of the Lotka-Voltera equations near its equilibrium points in the case $a < -1$, $b < -1$. Use the fact that for two-dimensional systems the phase portraits near the equilibrium points are approximated by those of the linearization, provided that the eigenvalues do not lie on the imaginery exis. [12]

   (b) Conjecture what the full phase portrait looks like. [6]

7. For the Lorenz system

$$dx/dt = \sigma(y - x), \quad dy/dt = rx - y - zx, \quad dz/dt = -\beta z + xy$$

   (a) Find all the steady state solutions to the system, stating clearly for what ranges of the parameters the solutions exist.

   (b) Linearize the system about its null solution and find the corresponding eigenvalues. Deduce that the trivial fixed point $(0, 0)$ is unstable for $r > 1$.

   (c) Using the Routh-Hurwitz criterion or otherwise, determine the stability of the nontrivial fixed points.

8. Consider the planar system

$$\dot{x} = \mu x - \omega y + (\alpha x - \beta y)[x^2 + y^2]$$

$$\dot{y} = \omega x + \mu y + (\beta x + \alpha y)[x^2 + y^2]$$

   where x and y are the state variables and μ is the control parameter.

   (a) For the linearized system, show that the eigenvalues of the Jacobian matrix for the fixed point $(0, 0)$ are $\lambda_1 = \mu - i\omega$, $\lambda_2 = \mu + i\omega$.

   (b) Show that a Hopf bifurcation of the fixed point $(0, 0)$ occurs when $\mu = 0$.

   (c) By using the transformation $x = r\cos\theta$, $y = r\sin\theta$ show that the system of equations transform into

$$\dot{r} = \mu r + \alpha r^3, \quad \dot{\theta} = \omega + \beta r^2$$

   (d) When $\alpha = -1$ and $\alpha = 1$, discuss the bifurcation that occurs at $\mu = 0$.

9. Given a system of differential equations

$$\dot{x} = x[(1 - x)(x - x_c) - \alpha y]$$
$$\dot{y} = ry[1 - \beta x - y]$$

where α, r, $\beta > 0$ and $-1 < x_c < 1$.

Find the equilibrium points of this system which satisfy $x \geq 0$, $y \geq 0$, $0 < x_c < 1$, $0 < \beta < 1$ noting carefully the parameter ranges for which each exists and discuss their stability. **[18]**

10. Consider the discrete predator-prey model

$$x_{n+1} = rx_n(1 - x_n) - x_n y_n, \qquad y_{n+1} = bx_n y_n$$

(a) Give an interpretation of each term in the equation. **[3]**

(b) Find the fixed points of these equations for $x \geq 0$, $y \geq 0$. **[3]**

(c) Determine the parameter ranges for which each of the fixed points is stable. **[12]**