NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 4135

DEPARTMENT OF APPLIED MATHEMATICS

SMA 4135 DYNAMICAL SYSTEMS

JUNE 2004 EXAMINATIONS

Time : 3 hours

Candidates must attempt ALL questions in section A and any TWO questions in section B.

SECTION A: Answer ALL questions in this section [50].

1. Define

   (a) an equilibrium point $\bar{x}$ of a differential equation $\dot{x} = f(x)$ where $f : \mathbb{R} \to \mathbb{R}$, and [2]

   (b) a fixed point $\bar{x}$ of a map $x_{n+1} = g(x_n)$ where where $g : \mathbb{R} \to \mathbb{R}$ [2]

   (c) Given that f and g are both continuously differentiable, state, without proof, sufficient conditions for the linear stability of the equilibrium point in (a) and the fixed point in (b). [2]

2. Consider the differential equation

$$\dot{x} = a^2 - x^2, \quad a \neq 0$$

   (a) Write down the equilibrium points of this differential equation [2]

   (b) Show that for $t \geq 0$,

$$x(t) = a\left[\frac{a + x(0) - [a - x(0)]e^{-2at}}{a + x(0) + [a - x(0)]e^{-2at}}\right].$$

[3]

   (c) Hence investigate the global asymptotic stability of the equilibrium points. [2]

3. Consider the second order linear equation

$$\ddot{x} + b\dot{x} + cx = 0 \tag{1}$$

Suppose that λ_1 and λ_2 are roots of the polynomial $y^2 + by + c = 0$.

(a) Show that the solution of (1) has the form

$$x(t) = \alpha e^{\lambda_1 t} + \beta e^{\lambda_2 t}$$

where α and β are determined by the initial conditions. [2]

(b) Obtain a matrix $\mathbf{P}$ such that

$$\begin{bmatrix} x(0) \\ \dot{x}(0) \end{bmatrix} = \mathbf{P} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

and hence compute α and β. [4]

(c) Find a matrix $\mathbf{A}$ such that (1) is equivalent to

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}$$

where $y = \dot{x}$. [2]

(d) Compute $\mathbf{P}^{-1}\mathbf{B}\mathbf{P}$ and $e^{\mathbf{A}t}$. [4]

4. (a) Define the term *topological conjugacy.* [2]

(b) Show that topological conjugacy preserves fixed points and periodic orbits for maps. [4]

(c) State the Hartman-GroBman theorem. [2]

(d) Let A be a real 2×2 matrix with complex eigenvalues $\alpha \pm i\beta$. Prove that there exists a real nonsingular matrix P such that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

[7]

5. Consider the logistic equation system

$$x_{n+1} = \lambda x_n(1 - x_n), \quad \text{for} \quad 0 \leq x_n \leq 1, \quad \text{and} \quad 0 \leq \lambda \leq 4.$$

(a) Find all the fixed points and determine their stability. [2]

(b) Show that for the logistic map a period-2 orbit exists for $r > 3$. [4]

(c) Deteremine the range for which the period-2 orbit is stable and draw the bifurcation diagram. [4]

SECTION B: Answer TWO questions in this section [50].

6. (a) Suppose **A** is the matrix

$$\mathbf{A} = \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

(i) Compute $e^{(\mathbf{A}t)}$. [3]

(ii) Hence, or otherwise, find the general solution of the differential equations

$$\begin{cases} \dot{x} = -x \\ \dot{y} = x - y + z \\ \dot{z} = -z \end{cases}$$

that is, give $(x(t), y(t), z(t))$ as a function of $(x(0), y(0), z(0))$. [3]

(b) The following equations are a model for interaction between two populations, one of which shows depensation in its growth rate:

$$\begin{cases} \dot{x} = x[(1-x)(x-x_c) - \alpha y] \\ \dot{y} = ry[1 - \beta x - y] \end{cases}$$

where $\alpha, \ \beta, \ r > 0$ and $-1 < x_c < 1$.

(i) Find the equilibrium points of this system which satisfy

$$x \geq 0, \quad y \geq 0, \quad 0 < x_c < 1, \quad 0 < \beta < 1$$

noting carefully the parameter ranges for which each exists. [5]

(ii) Discuss the stability of each of these equilibrium points. [8]

(iii) Sketch phase portraits for these equations, near equilibrium points for

$$0 < x_c < 1, \quad 0 < \beta < 1.$$

Use the fact that for two-dimensional systems the phase portraits near the equilibrium points are approximated by those of the linerization, provided that the eigenvalues do not lie on the imaginery axis. [3]

(iv) Conjecture what the full phase portrait looks like for

$$0 < x_c < 1, \quad 0 < \beta < 1.$$

[3]

7. (a) Consider the following dynamical system

$$\begin{cases} \dot{x} = x(a - x^2 - y^2) - y \\ \dot{y} = y(a - x^2 - y^2) + x \end{cases}$$

(i) Show that the system has a fixed point at the origin and classify it. Verify that for $a > 0$ there are periodic solutions of the form

$$(x(t), y(t)) = (\sqrt{a}\cos t, \sqrt{a}\sin t)$$

[5]

(ii) State Bendixson criterion for the existence of periodic orbits and use the criterion to determine conditions for the existence of periodic solutions in the given system. [3]

(iii) Transform the system into polar coordinates using

$$x = r\cos\theta, \qquad y = r\sin\theta$$

From the equation for $\dot{\theta}$ conclude that the fixed point at the origin is the only one. [2]

(iv) Show that the system has a periodic orbit determine the equation of the orbit. [2]

(v) Plot also the bifurction diagram. [2]

(b) Consider the dynamics on $\mathbb{R}^m$ given by

$$\mathbf{x}_{n+1} = \mathbf{B}\mathbf{x}_n + \mathbf{b}$$

where $\mathbf{B}$ ian an $n \times n$ matrix and $\mathbf{b} \in \mathbb{R}^n$ is a fixed vector.

(i) If $\mathbf{B}$ has no eigenvalues $\lambda = 1$, find a coordinate change of the form $\mathbf{y} = \mathbf{x} - \mathbf{c}$ for a fixed $\mathbf{c} \in \mathbb{R}^n$ such that in the new coordinate system the dynamics is given by

$$\mathbf{y}_{n+1} = \mathbf{B}\mathbf{y}_n.$$

[4]

(ii) Hence, or otherwise, compute the general orbit (u_n, v_n) of the map

$$\begin{cases} u_{n+1} = 4u_n + 2v_n + 3 \\ v_{n+1} = -u_n - v_n + 4 \end{cases}$$

that is, give (u_n, v_n) as a function of (u_0, v_0). [6]

8. (a) The classical Lotka-Volterra equations describing the time evolution of a prey-predator ecosystem can be written as:

$$\begin{cases} \dot{x} = ax - xy \\ \dot{y} = xy - by. \end{cases}$$

(i) Linearize these for small variations about the trivial solution $x = y = 0$ and determine the eigenvalues. [2]

(ii) For the initial conditions $x = x_0,\ y = y_0$, determine the equation the equation of the trajectory, $y(x)$, in phase space that is predicted by the linearized equations. [3]

(iii) Sketch this trajectory for case of $a = b = 1$. [2]

(b) For the Lorenz system

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = rx - y - xz \\ \dot{z} = -\beta z + xy. \end{cases}$$

(i) Find all steady state solutions to the system, stating clearly for what ranges of the parameters the solutions exist. [4]

(ii) Linearize the system about its null solution and find the corresponding eigenvalues. Deduce that the trivial fixed point $(0, 0, 0)$ is unstable for $r > 0$. What sort of point is the orgin when it is unstable? [6]

(iii) Given a Lyapunov function of the form

$$H(\mathbf{x}) = \frac{1}{2}[x^2 + \sigma y^2 + \sigma z^2],$$

use the direct method to show that if $r < 1$ then the null solution is globally stable. [4]

(iv) Use the Routh-Hurwirz criterion or otherwise, to determine the stability of the nontrivial fixed points. [4]

9.

(a) Consider the discrete predator-prey model:

$$\begin{cases} x_{n+1} = rx_n(1 - x_n) - x_n y_n \\ y_{n+1} = bx_n y_n \end{cases}$$

(i) Give an interpretation of each term in the equation. [4]

(ii) Find the fixed points of these equations for $x, y \geq 0$. [4]

(iii) Determine the parameter ranges for which each of the fixed points is stable. [5]

(b) Consider the Van der Pol Oscillator:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -kx - (x^2 - \lambda)y \end{cases}$$

where k is a positive constant.

(i) Show that for all values of a, a Hopf bifurcation occurs from the equilibrium point at $x = 0 = y$ at a value of a which you can determine and compute the direction of branching of the periodic solution that appears in the Hopf bifurcation. [8]

(ii) Sketch phase portraits of this system of equations, for the two cases $\lambda < 0$ and $\lambda > 0$. [4]

Hint: For (b) you may use the fact that the Hopf bifurcation which takes place in the equation :

$$\dot{z} = (\lambda + i\omega)z + a_{20}z^2 + a_{11}z\bar{z} + a_{02}\bar{z}^2 + a_{30}z^3 + a_{21}z^2\bar{z} + a_{12}z\bar{z}^2 + a_{03}\bar{z}^3 + \ldots$$

is supercritical if $a = \frac{1}{\omega}\mathrm{Im}\{a_{20}a_{11}\} + \mathrm{Re}\{a_{21}\}$ is negative and subcritical if the sign is reversed.

END OF QUESTION PAPER