NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 4135

DEPARTMENT OF APPLIED MATHEMATICS

SMA 4135 DYNAMICAL SYSTEMS

DECEMBER 2004 EXAMINATIONS

Time : 3 hours

Candidates must attempt ALL questions in section A and any TWO questions in section B.

SECTION A: Answer ALL questions in this section [50].

1. (a) Define
   (i) an equilibrium point $\bar{x}$ of a differential equation $\dot{x} = f(x)$ where $f : \mathrm{R} \to \mathrm{R}$, [2]
   (ii) a fixed point $\bar{x}$ of a map $x_{n+1} = g(x_n)$ where where $g : \mathrm{R} \to \mathrm{R}$. [2]

   (b) Given that f and g are both continuously differentiable, state, without proof, sufficient conditions for the linear stability of the equilibrium point in (a) and the fixed point in (b). [2]

2. Consider the differential equation

$$\dot{x} = a(x-b)(x-c), \quad \text{where} \quad b < c \quad \text{and} \quad x(0) = x_0.$$

   (a) Write down the equilibrium points of this differential equation. [2]

   (b) Sketch the phase portrait. For each initial condition x_0, use this qualitative information to determine the long-term behaviour of the solutions. Treat the cases $a < 0$, $a = 0$, and $a > 0$. [2]

   (c) Solve the initial value problem. Verify that the behaviour of this solution is consistent with the qualitative information in (a). [2]

   (d) Show that if $a > 0$ and $x_0 > c$, or if $a < 0$ and $x_0 < b$, the solutions blow up in finite time. Find the largest interval $[0, T]$ on which the solutione exists. What happens if $b = c$? [2]

3. Consider the second order linear equation

$$\ddot{x} + b\dot{x} + cx = 0, \tag{1}$$

Suppose that the polynomial $y^2 + by + c = 0$ has a complex pair of eigenvalues $\mu \pm i\omega$.

(a) Show that the solution of (1) has the form

$$x(t) = (\alpha \cos \omega t + \beta \sin \omega t)e^{\mu t}$$

where α and β are determined by the initial conditions. [2]

(b) Obtain a matrix **P** such that

$$\begin{bmatrix} x(0) \\ \dot{x}(0) \end{bmatrix} = \mathbf{P} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

and hence compute α and β. [4]

(c) Find a matrix **A** such that (1) is equivalent to

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

where $y = \dot{x}$. [2]

(d) Compute $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ and $e^{\mathbf{A}t}$. [3]

4. (a) Define the term *topological conjugacy.* [2]

(b) Show that topological conjugacy preserves fixed points and periodic orbits for maps. [4]

(c) State the Hartman-Grobman theorem. [2]

(d) Let A be a real 2×2 matrix with complex eigenvalues $\alpha \pm i\beta$. Prove that there exists a real nonsingular matrix P such that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

[7]

5. Consider the logistic equation system

$$x_{n+1} = (1 + \lambda)x_n - (\lambda/K)x_n^2.$$

(a) Find all the fixed points and determine their stability. [2]

(b) Show that for the logistic map a period-2 orbit exists for $\lambda > 2$. [4]

(c) Deteremine the range for which the period-2 orbit is stable and draw the bifurcation diagram. [4]

SECTION B: Answer TWO questions in this section [50].

6. (a) Suppose **A** is the matrix

$$\mathbf{A} = \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

(i) Compute $e^{(\mathbf{A}t)}$. [3]

(ii) Hence, or otherwise, find the general solution of the differential equations

$$\begin{cases} \dot{x} = -x \\ \dot{y} = x - y + z \\ \dot{z} = -z, \end{cases}$$

that is, give $(x(t), y(t), z(t))$ as a function of $(x(0), y(0), z(0))$. [3]

(b) The following equations are a model for interaction between two populations:

$$\begin{cases} \frac{dN}{dT} = rN(1 - \frac{N}{K}) - \frac{cNP}{a+N}, \\ \frac{dP}{dT} = \frac{bNP}{a+N} - mP. \end{cases}$$

(i) By introducing the change of variables

$$N = ax, \quad P = r\frac{a}{c}y, \quad T = \frac{t}{r}$$

show that this system reduces to

$$\begin{cases} \frac{dx}{dt} = x(1 - \frac{x}{\gamma}) - \frac{xy}{1+x}, \\ \frac{dy}{dt} = \beta(\frac{x}{1+x} - \alpha)y, \end{cases}$$

where

$$\alpha = \frac{m}{b}, \quad \beta = \frac{b}{r}, \quad \gamma = \frac{K}{a}.$$

[3]

(ii) Find the equilibrium points of this system noting carefully the parameter ranges for which each exists. [5]

(iii) Discuss the stability of each of these equilibrium points. [8]

(iv) Sketch phase portraits for these equations, near equilibrium point $[(x*, y*) = (x*, (1 + x*)(1 - x * /\gamma]$ where $x* = \alpha/(1 - \alpha)$. [3]

7. (a) State Bendixson's criterion for existence of periodic solutions and prove it. [3]

(b) Consider the following dynamical system

$$\begin{cases} \dot{x} = \mu x - \omega y - x(x^2 + y^2) \\ \dot{y} = \omega x + \mu y - y(x^2 + y^2). \end{cases}$$

(i) Show that the system has a fixed point at the origin and classify it. [5]

(ii) Transform the system into polar coordinates using

$$x = r\cos\theta, \qquad y = r\sin\theta\ .$$

From the equation for $\dot{r}$ conclude that the fixed point at the origin is the only one. [2]

(iii) Show that the system has a periodic orbit determine the equation of the orbit. [2]

(iv) Plot the bifurction diagram. [2]

(c) Consider the dynamics on R^m given by

$$\mathbf{x}_{n+1} = \mathbf{B}\mathbf{x}_n + \mathbf{b}$$

where $\mathbf{B}$ ian an $n \times n$ matrix and $\mathbf{b} \in \mathrm{R}^n$ is a fixed vector.

(i) If $\mathbf{B}$ has no eigenvalues $\lambda = 1$, find a coordinate change of the form $\mathbf{y} = \mathbf{x} - \mathbf{c}$ for a fixed $\mathbf{c} \in \mathrm{R}^n$ such that in the new coordinate system the dynamics is given by

$$\mathbf{y}_{n+1} = \mathbf{B}\mathbf{y}_n.$$

[4]

(ii) Hence, or otherwise, compute the general orbit (u_n, v_n) of the map

$$\begin{cases} u_{n+1} = 4u_n + 2v_n + 3 \\ v_{n+1} = -u_n - v_n + 4 \end{cases}$$

that is, give (u_n, v_n) as a function of (u_0, v_0). [6]

8. (a) The classical Lotka-Volterra equations describing the time evolution of a prey-predator ecosystem can be written as:

$$\begin{cases} \dot{x} = ax - xy \\ \dot{y} = xy - by. \end{cases}$$

(i) Linearize these for small variations about the trivial solution $x = y = 0$ and determine the eigenvalues. [2]

(ii) For the initial conditions $x = x_0,\ \ y = y_0$, determine the equation of the trajectory, $y(x)$, in phase space that is predicted by the linearized equations. [3]

(iii) Sketch this trajectory for case of $a = b = 1$. [2]

(b) Consider the endemic SIR Model

$$\begin{cases} dS/dt = \mu N - \mu S - \beta IS/N) \\ dI/dt = \beta IS/N - \gamma I - \mu I \\ dR/dt = \gamma I - \mu R \end{cases}$$

where the population size N is divided into susceptibles $S(t)$, infectives $I(t)$ and recovered such that $S(t) + I(t) + R(t) = N$.

(i) Briefly discuss the model assumptions which were made to create this model. [4]

(ii) By dividing the equations by a constant N, show that the SIR model above can be collapsed into a two population problem

$$\begin{cases} ds/dt = \mu - \mu s - \beta is \\ di/dt = \beta is - (\gamma + \mu)i \end{cases}$$

where $s = S/N$, $i = I/N$, $r = R/N$, and $s + i + r = 1$.

(iii) Find all steady state solutions to the system, stating clearly for what ranges of the parameters the solutions exist. [6]

(iv) Determine the stability of the disease-free equibrium and determine the condition for an epidemic to occur if a single infectious individual is introduced into the polulation. [6]

(v) Sketch the bifurcation diagram indicating the transition from the non-endemic to the endemic state. [6]

9.

(a) Consider the discrete predator-prey model:

$$\begin{cases} x_{n+1} = (1+r)x_n - rx_n^2 - ax_ny_n \\ y_{n+1} = bx_ny_n \end{cases}$$

(i) Give an interpretation of each term in the equation. [4]

(ii) Find the fixed points of these equations for $x, y \geq 0$. [4]

(iii) Determine the parameter ranges for which each of the fixed points is stable. [5]

(b) Consider the Van der Pol Oscillator:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -kx - (x^2 - \lambda)y \end{cases}$$

where k is a positive constant.

(i) Show that for all values of a, a Hopf bifurcation occurs from the equilibrium point at $x = 0 = y$ at a value of a which you can determine and compute the direction of branching of the periodic solution that appears in the Hopf bifurcation. [8]

(ii) Sketch phase portraits of this system of equations, for the two cases $\lambda < 0$ and $\lambda > 0$. [4]

Hint: For (b) you may use the fact that the Hopf bifurcation which takes place in the equation :

$$\dot{z} = (\lambda + i\omega)z + a_{20}z^2 + a_{11}z\bar{z} + a_{02}\bar{z}^2 + a_{30}z^3 + a_{21}z^2\bar{z} + a_{12}z\bar{z}^2 + a_{03}\bar{z}^3 + \ldots.$$

is supercritical if $a = \frac{1}{\omega}\text{Im}\{a_{20}a_{11}\} + \text{Re}\{a_{21}\}$ is negative and subcritical if the sign is reversed.

END OF QUESTION PAPER