NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF APPLIED MATHEMATICS

SMA 4151 Statistical Inference

May/June 2004 Time: 3 hours

Answer **ALL** questions in Section A and **ANY THREE** from section B

Section A [40 marks] Answer all questions from this section

A1. Let $X_1,\ldots,X_n$ be a random sample from a log-normal distribution with parameters μ and σ^2 (where σ^2 is known), i.e.

$$f(x;\mu;\sigma^2)=\frac{1}{x\sigma\sqrt{2\pi}}e\left[^{-\frac{1}{2}(\frac{\ln x-\mu}{\sigma})^2}\right],0<x.$$

(a) Find the method-of-moments estimator of θ.
(b) Find a maximum-likelihood estimator of θ.
(c) Find the Cramer-Rao lower bound of the variance of unbiased estimators of θ.

[13 marks]

A2. (a) State the Neymann-Pearson Lemma.

(b) Let $X_1,\ldots,X_n$ be a random sample from the Pareto density:

$$f(x;\theta)=\frac{\tau^\theta\theta}{x^{\theta+1}}\mathrm{I}_{(\tau,\infty)}(x) \text{ where } \tau>0 \text{ is known.}$$

(i) Show that if $Y=\ln\left(\frac{X}{\tau}\right)$, then $Y\sim Exponential(\theta)$.

(ii) Find an α-size most powerful test for testing the hypothesis $H_o:\theta=\theta_o$ versus $H_1:\theta=\theta_1$ where $\theta_o<\theta_1$.

[11 marks]

A3. Let X be a single observation from the density $f(x;\theta)=\theta x^{\theta-1}\mathrm{I}_{(0,1)}(x)$, where $\theta>0$. Find a pivotal quantity and use it to find a confidence interval for θ. **[5marks]**

A4. (a) Let $\theta\in\Theta$ be a parameter of a population with probability density $f(x;\theta)$ from which a random sample $X_1,\ldots,X_n$ has been obtained. Suppose two hypotheses concerning θ, $H_o:\theta\in\Theta_o$ versus $H_1:\theta\notin\Theta_o$ are to be tested using the test γ^*. State the two conditions that are necessary and sufficient for γ^* to be a uniformly most powerful test.

(b) What do you understand by the **generalised likelihood ratio test** principle?

(c) Let $X_1,\ldots,X_n$ be a random sample from a population that follows a beta distribution, i.e.

$$f(x;a,b)=\frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}\mathrm{I}_{(0,1)}(x).$$

Let γ^* : reject H_o iff $\sum_{i=1}^{n}\ln x_i > k(\alpha)$ (where $k(\alpha)$ is a constant dependent on the level of significance desired) be a test of hypothesis $H_o: a=a_o$ versus $H_1: a>a_o$. Show that γ^* is a uniformly most powerful size -α test for testing $H_o: a=a_o$ versus $H_1: a>a_o$. You are to assume that b is known.

[11 marks]

SECTION B (60 marks)

Answer any **THREE** questions from this section

B5. (a) Suppose a random sample $X_1,\ldots,X_n$ is drawn from a population whose density has parametric vector $\underline{\theta}$. Define the following terms

(i) uniformly minimum variance unbiased estimator (UMVUE) of a function of a parameter θ, $\tau(\underline{\theta})$.

(ii) Minimal sufficient set of statistics

(b) Let $X_1,\ldots,X_n$ be a random sample from the density

$$f(x;\theta)=\theta(1+x)^{-(1+\theta)}\mathrm{I}_{(0,\infty)}(x) \text{ where } \theta>0.$$

(i) Estimate θ by the method of moments assuming $\theta>1$.

(ii) Find the maximum likelihood estimator of $\frac{1}{\theta}$.

(iii) Find the Cramer-Rao lower bound for the variance of unbiased estimators of $\frac{1}{\theta}$.

[20 marks]

B6. (a) Briefly explain the purpose of the subject "Statistical Inference".

(b) Briefly outline the advantages of interval estimation over point estimation when estimating an unknown population parameter.

(c) Let $X_1,\ldots,X_n$ be a random sample obtained from a population which follows a normal distribution with parameters μ and σ^2. Let

$$\overline{X}=\frac{1}{n}\sum_{i=1}^{n}X_i \text{ and } S_n^{\,2}=\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\overline{X})^2.$$

Prove that

(i) $S_n^{\,2}$ is an unbiased estimator of σ^2.

(ii) The sequence $\{S_n^{\,2}\}$ is a mean-squared error consistent sequence of estimators of σ^2.

[20marks]

B7. (a) Let p denote the number of successes occurring in a Poisson process in a 5-minute period. Calculate α and β if you wish to test $H_o : \mu = 2$ against $H_1 : \mu = 3$ where μ is the mean for a 5-minute period and $x > 4$ is used as the critical region.

(b) Let $X_i, i = 1,...,100$ be $Bin(n, p)$. Let $Y = \sum_{i=1}^{100} X_i$ be the number of successes in 100 independent trials. Let the hypothesis to be tested be $H_o : p = p_o$ against $H_1 : p = p_1$. Let the test γ be defined by the critical region $c = [|y - 50| > 10]$. Find and sketch the power function for this test.

[20marks]

B8. (a) A sport fishing boat takes 4 people fishing each day, during the salmon season. The ticket seller tells each prospective customer that his probability of catching at least one salmon each day exceeds 0.6. On 5 successive days the boat took people fishing. The number who caught at least one salmon, on these 5 days were 3, 1, 2, 4 and 3 respectively. Assume that for each day the persons on the boat are independent Bernoulli trials with probability θ of catching at least one salmon. Using $\alpha \leq 0.05$ and a likelihood ratio test, would you accept the ticket seller's claim?

(b) If $X_1,..., X_n$ are from the density function $f(x, \theta) = \theta e^{-\theta x}, x > 0$.

(i) Find the likelihood function.

(ii) Find the likelihood ratio test for the hypothesis $H_o : \theta = \theta_o$ against $H_1 : \theta = \theta_1 < \theta_o$ and design an α-size likelihood ratio test.

[20marks]

END OF EXAMINATION