NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY
SMA 4162

DEPARTMENT OF APPLIED MATHEMATICS

BSc Honours in Applied Mathematics: Part IV

Numerical Methods For Differential Equations.

DECEMBER 2004

Time : 3 hours

Answer any **FOUR** questions being carefully to number them Q1 - Q6. All questions carry **equal** marks

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Q1. (a) Derive composite Trapezium Rule

$$\int_a^b f(x)dx \approx \frac{1}{2}h\big[f_0 + 2f_1 + 2f_2 + ... + 2f_{n-1} + f_n\big]$$

and show that the associated truncation error is

$$E_{\text{trunc}} = -\frac{1}{12}(b-a)h^2 f^{(2)}(c).$$

[11]

(b) Consider $f(x) = 2 + \sin(2\sqrt{x})$.

(i) Show that the exact value of the definite integral $\int_1^6 2 + \sin(2\sqrt{x})dx$ is $2x - \sqrt{x} - \cos(2\sqrt{x}) + \sin 2\sqrt{x}$. [5]

(ii) Investigate the error when the composite trapezoidal rule is used over $[1, 6]$ with $h = 0.5$. [6]

(iii) Given that when $h = 0.25, \quad 0.125, \quad 0.0625$ the values of the intergral are:

$$8.18604926; \quad 8.18412019; \quad 8.18363936$$

respectively show that when h is reduced by a factor of $\frac{1}{2}$ the successive errors are diminished by approximately $\frac{1}{4}$. [3]

Q2. Consider the boundary value problem

$$y'' + xy + y = 1, \quad 0 \le x \le 1, \quad y(0) = 0, \quad y(1) = 1.$$

(a) Derive the central difference formula for the first and second derivative and the associated errors of $O(h^2)$. [6]

(b) Derive the corresponding finite difference equation using a central difference formula for N intervals of width h where $h = \frac{1}{N}$. [4]

(c) Solve the finite difference equations for $h = 0.25$. [9]

(d) The boundary value problem can be solved using the shooting method write down the algorithm of this method. [6]

Q3. (a) Consider the initial-value problem

$$y'' = y^2 - x + xy; \quad 0 \le x \le 1, \quad y(0) = 1; \quad y'(0) = 1.$$

(i) Reduce the above differential equation to a system of two equations. [2]

(ii) Use Euler's method to solve the equation at $y(0.6)$ with $h = 0.2$. [6]

(iii) The the initial-value problem is changed into a boundary value problem

$$y'' = y^2 - x + xy; \quad 0 \le x \le 1, \quad y(0) = 1; \quad y(1) = 3.$$

In using the shooting method and Euler estimation with $h = 0.2$ on the problem the following results were obtained:

$$y'(0) = 1 \Rightarrow y(1) = 2$$

$$y'(0) = 2 \Rightarrow y(1) = 3.5.$$

Show that the value of $y'(0)$ to be used on the next 'shot' is $\frac{5}{3}$. [3]

(b) Figure 1 shows the right cross-section of a thick tube and the enlargement of a quarter of it. The cross section is symmetrical about AA' and BB'. The stress function u satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2 = 0$$

and $u = 0$ on boundaries ABC and A'B'C'.

(i) Assuming the Taylor series expansion obtain a finite difference approximation to the stress function. [2]

(ii) Set up the finite difference scheme to find an approximate solution at the nodes u_1, u_2, u_3, u_4 and u_5 and $h = 0.05$. [6]

(iii) Solve, analytically, the finite difference equations. [6]

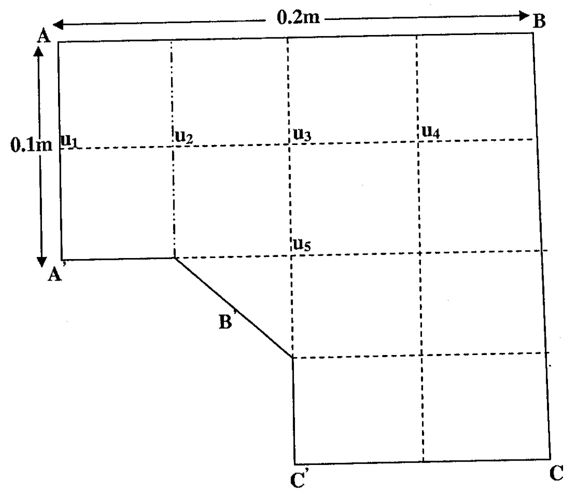

Figure 1: Right cross-section of a thick tube and the enlargement of a quarter of it.

Q4. (a) Consider the following Runge-Kutta m-order method for solving the differential equation $y' = f'(x,y)$:

$$y(x+h) + y(x) = h\sum_{s=1}^{m} w_s k_s$$

where

$$k_1 = f(x,y), ...k_s = f\Big(x + a_s h, y + h\sum_{r=1}^{s-1}\beta_{sr}k_r\Big).$$

(i) Using Taylor's series show that the above equation can be written as

$$y(x+h)+y(x) = hf+\frac{h^2}{2}(f_x+ff_y)+\frac{h^3}{6}(f_{xx}+2ff_{xy}+f_xf_y+f^2f_{yy}+ff_y^2+o(h^4))$$

where $f = f(x,y)$. [5]

(ii) Hence derive Runge-Kutta's second order method. [6]

(iii) Consider the initial value problem

$$y' = y - x \qquad y(0) = 2$$

on the interval [0,1] with $h = 0.1$.

(a) Use Runge-Kuta's fourth order method to solve the differential equation at $x = 0.1; x = 0.2; = 0.3$ to 7 significant figures. [6]

(b) Use the Adams-Bashforth-Moulton method to solve the differential equation at $x = 0.4$ to 7 significant figures. [6]

(c) Given that $y(0.4) = 2.8918242$ from Runge-Kuta's fourth order method, compare the two methods used in this case. [2]

Q5. (a) Show how

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le l$$

becomes, using a suitable change of variables,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le 1.$$

[3]

(b) Consider $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 \le x \le 1$; with boundary conditions

$$u(0,t) = 100, \quad \frac{\partial u}{\partial x}(1,t) = 0$$

and initial condition $u(x,0) = 50$ for $0 \le x \le 1$. Derive the finite difference scheme using the explicit method and show that it reduces to

$$\mathbf{U}_{j+1} = A\mathbf{U}_{\mathrm{j}}.$$

[5]

(i) For $t = 0.02$ and step width $h = 0.2$ and $\delta t = 0.01$

(a) solve the equation by the explicit method. [6]

(b) show that after s stages the error, e, becomes $A^s e$. [5]

(ii) Form the equations (but do not solve) for the temperatures given by the Crank-Nicholson method. [6]

Q6. A conducting rod AB, of unit length, has a prescribed temperature at the end A and loses heat through convection at B. Expressed in variational form the temperature u satisfies: $u = 100$ at A and minimses

$$V(u) = \int_0^1 \frac{1}{2}\left(\frac{du}{dx}\right)^2 dx + (u_B - 40)^2$$

where u_B is the unknown temperature at B.

(a) Derive the stiffness matrix and force vector for a two nodes linear element. [8]

(b) Hence form the finite element equations for four equally sized elements. [7]

(c) Obtain the solution of the finite element equations. [5]

(d) Show, using graphs, the temperature and heat flow $q = -(\frac{du}{dx})$. [5]

END OF QUESTION PAPER