NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 4162

DEPARTMENT OF APPLIED MATHEMATICS

BSc Honours in Applied Mathematics: Part IV

Numerical Methods For Differential Equations.

SUPPLEMENTARY EXAMINATION JULY 2005

Time : 3 hours

Answer any **FOUR** questions being carefully to number them Q1 - Q6. All questions carry equal marks

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Q1. Consider the initial value problem

$$y' = (1+x)y \qquad y(0) = 1$$

(a) Use Euler's method with $h = 0.1$ to generate a solution of the initial value problem, correct to three decimal places, when $x = 0.2$. [4]

(b) Use the Taylor Series Method to show that for a step h,

$$y(h) = 1 + h + h^2 + \frac{2}{3}h^3 + ...$$

Obtain $y(0.2)$ accurate to three decimal places. [6]

(c) Show that the Runge-Kutta second order method, when simplified, gives the corresponding value of (b) as

$$y(h) = 1 + h + h^2 + \frac{1}{2}h^3.$$

Obtain $y(0.2)$ accurate to three decimal places. [6]

(d) Given that the exact value of $y(0.2)$, correct yo three decimal places, is 1.246, compare the answers from the three methods used. [3]

(e) Derive the central difference formula and the associated error of $O(h^2)$. [6]

Q2. (a) Use Simpson's rule to evaluate the integral

$$\int_0^{\frac{\pi}{2}} (1-\sin^3 x)^5 dx$$

[5]

(b) Consider the boundary-value problem

$$y'' + xy' + y = 1; \quad 0 < x < 1, \quad y(0) = 1; y(1) = 1.$$

(i) In using the shooting method and Euler estimation with $h = 0.25$ on the problem the following results were obtained:

$$y'(0) = 0 \Rightarrow y(1) = 0.3481$$

$$y'(0) = 0.5 \Rightarrow y(1) = 0.7876.$$

(a) Show that the value of $y'(0)$ to be used on the next 'shot' is 0.7416. [3]

(b) Using the initial condition $y'(0) = 0.7416$ find the solution of the boundary value-problem. [5]

(ii) Use the finite difference method with five nodes to obtain a second approximate solution. [10]

(iii) Compare the two methods. [2]

Q3. (a) Consider the initial value problem

$$y' = 1 - x + 4y \qquad y(0) = 1$$

on the interval [0,1] with $h = 0.1$.

(i) Use Runge-Kutta's fourth order method to solve the differential equation at $x = 0.1$; $x = 0.2$; $x = 0.3$. [8]

(ii) Explain Adams-Bashforth-Moulton method and write down its algorithm. [4]

(iii) Use the Adams-Bashforth-Moulton method to solve the differential equation at $x = 0.4$. [8]

(iv) How does the correction formula in (iii) reduce the error in y_4? [5]

Q4. Consider the parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq 1, \quad u(0) = 0, \quad \text{on} \quad x = 0, x = 1$$

and

$$u = x(1-x) \quad when \quad t = 0.$$

(a) Set up the explicit finite difference scheme to solve the equation. [3]

(b) estimate u at $t = 0.04$ using as a time step size $deltat = 0.02$. [9]

(c) Derive the Crank-Nicholson scheme to solve this problem, that is,

$$(2+2r)u_{ij+1} = ru_{i-1j} + (2-2r)u_{ij} + ru_{i+1j} + ru_{i-1j+1} + ru_{i+1j+1}.$$

[4]

(d) Use 4 initial nodes with $h = 0.2$, and one time step to estimate u at $t = 0.2$. [9]

Q5. Consider the boundary value problem

$$y''(x) - y(x) = 1, \quad 0 < x < 1, \quad y(0) = 0, \quad y(1) = 0.$$

(a) Show that the analytic solution is

$$y(x) = \frac{1}{e+1}\left(e^x + e^{1-x}\right) - 1.$$

[4]

(b) Derive the corresponding finite difference using a central difference formula for N intervals of width h where $h = \frac{1}{N}$. [4]

(c) Solve the difference equations for $h = 0.2$. [8]

(d) Tabulate the errors and verify that they behave as $O(h^2)$. [4]

(e) Write the algorithm of the Jordan-Seidel iteration method to solve the resultant system. [5]

Q6. A conducting rod AB, of unit length, has a prescribed temperature at the end A and loses heat through convection at B. Expressed in variational form the temperature u satisfies: $u = 100$ at A and minimses

$$V(u) = \int_0^1 \frac{1}{2}\left(\frac{du}{dx}\right)^2 dx + (u_B - 40)^2$$

where u_B is the unknown temperature at B.

(a) Derive the stiffness matrix and force vector for a two nodes linear element. [8]

(b) Hence form the finite element equations for four equally sized elements. [7]

(c) Obtain the solution of the finite element equations. [5]

(d) Show, using graphs, the temperature and heat flow $q = -(\frac{du}{dx})$. [5]

END OF QUESTION PAPER