NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 4162

DEPARTMENT OF APPLIED MATHEMATICS

BSc Honours in Applied Mathematics: Part IV

Numerical Methods For Differential Equations.

DECEMBER 2005

Time : 3 hours

Answer any **FOUR** questions being carefully to number them Q1 - Q6. All questions carry equal marks

Q1. Consider the following Runge-Kutta m-order method for solving the differential equation $y' = f(x, y)$:

$$y(x+h) - y(x) = h\sum_{s=1}^{m} w_s k_s$$

where

$$k_1 = f(x, y), \ldots . k_s = f\Big(x + a_s h, y + h\sum_{r=1}^{s-1} \beta_{sr} k_r\Big).$$

(a) Describe how the Runge-Kutta methods work. [2]

(b) Using Taylor's series show that the above equation can be written as

$$y(x+h)+y(x) = hf + \frac{h^2}{2}(f_x + ff_y) + \frac{h^3}{6}(f_{xx} + 2ff_{xy} + f_x f_y + f^2 f_{yy} + ff_y^2 + o(h^4))$$

where $f = f(x, y)$. [5]

(c) Hence derive Runge-Kutta's second order method. [6]

(d) Show that Runge's third order method agrees with the Taylor series method of the same order for the differential equation

$$y' = y + x.$$

[7]

(e) Write a short computer program in C to solve the initial-value problem

$$y' = y + x, y(0) = 1$$

using the second order Runge-Kutta method. [5]

Q2. Consider the initial value problem

$$y' = y^2 + 1 \qquad y(0) = 0$$

(a) Use Modified Euler's method with $h = 0.1$ to generate a solution of the initial value problem, correct to seven decimal places, when $x = 0.4$. [6]

(b) Explain Adams-Bashforth-Moulton method. [3]

(c) Given that $y(0.1) = 0.1003346, y(0.2) = 0.2027099, y(0.3) = 0.3093360, y(0.4) = 0.4227930$, from the Runge-Kutta's fourth order method, use the Adams-Bashforth-Moulton method to solve the differential equation at $x = 0.4$ correct to seven decimal places. [6]

(d) If the exact value of $y(0.4)$, correct to seven decimal places, is 0.4227932, compare the answers from the three methods used. [4]

(e) How does the correction formula in (iii) reduce the error in $y(0.4)$? [2]

(f) Write a short computer program in C to solve the initial-value problem using the Adams-Bashforth-Moulton method. [4]

Q3. (a) Derive the central difference formula for the first and second derivatives and the associated errors. [6]

(b) Consider the boundary value problem

$$y''(x) - y(x) = 1, \quad 0 < x < 1, \quad y(0) = 0, \quad y(1) = 0.$$

(i) Show that the analytic solution is

$$y(x) = \frac{1}{e+1}\left(e^x + e^{1-x}\right) - 1.$$

[4]

(ii) Derive the corresponding finite difference using a central difference formula for N intervals of width h where $h = \frac{1}{N}$. [4]

(iii) Solve the difference equations for $h = 0.2$. [8]

(iv) Tabulate the errors and verify that they behave as $O(h^2)$. [3]

Q4. (a) Consider the initial-value problem

$$y'' - 3y' + 2y = 0; y(0) = -1; y'(0) = 0.$$

(i) Solve, analytically, the differential equation at $x = 0.1$. [3]

(ii) Reduce the above differential equation to a system of two equations. [2]

(iii) Use Runge-Kutta's fourth order method to solve the differential equation at $x = 0.1$ on the interval [0,1] with $h = 0.1$. [8]

(iv) Calculate the error at $y(0.1)$ [2]

(b) Consider the boundary-value problem

$$y'' + xy' + y = 1; \quad 0 < x < 1, \quad y(0) = 0; \quad y(1) = 1.$$

(i) Write down the algorithm of the shooting method. [4]

(ii) In using the shooting method and Euler estimation with $h = 0.25$ on the problem the following results were obtained:

$$y'(0) = 0 \Rightarrow y(1) = 0.3481$$

$$y'(0) = 0.5 \Rightarrow y(1) = 0.7876.$$

(a) Show that the value of $y'(0)$ to be used on the next 'shot' is 0.7416. [2]

(b) Using the initial condition $y'(0) = 0.7416$ find the solution of the boundary value-problem at $x = 0.1$. [4]

Q5. (a) Assuming the Taylor series expansion obtain a finite difference approximation to

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2u$$

over a square grid. [2]

(i) Set up the finite difference scheme to find an approximate solution over the square $0 \leq x \leq 3, \quad 0 \leq y \leq 4$ with $u = 2x^2 y$ on all the boundary, using $h = 1$. [5]

(ii) Solve the differential equation using two iterations of the Gauss-Seidel method. [6]

(b) Consider the parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq 1, \qquad u(0) = 0, \quad \text{on} \quad x = 0, x = 1$$

and

$$u = x(1 - x) \quad when \quad t = 0.$$

(i) Derive the Crank-Nicholson scheme to solve this problem, that is,

$$(2 + 2r)u_{ij+1} = ru_{i-1j} + (2 - 2r)u_{ij} + ru_{i+1j} + ru_{i-1j+1} + ru_{i+1j+1}.$$

[4]

(ii) Use 4 initial nodes with $h = 0.2$, and one time step to estimate u at $t = 0.2$. [8]

Q6. An insulating wall is constructed of three homogeneous layers of differing conductivity, $K_1 = 5, K_2 = 3, K_3 = 7$ and widths $w_1 = 0.05, w_2 = 0.06, w_3 = 0.02$. At the inner side A at the beginning of the wall, the temperature is given as $u^* = 50$ and at outer side B heat is lost through convection with $h = 50$ and $u_\infty = 20$ (see diagram below). Expressed in variational form u satisfies $u(0) = u^*$ and minimises the functional

$$V[u(x)] = \int_0^{0.13} \frac{1}{2} K u'^2 dx + \frac{h}{2} u^2(B) - h u_\infty u(B),$$

where K is the conductivity of the layer.

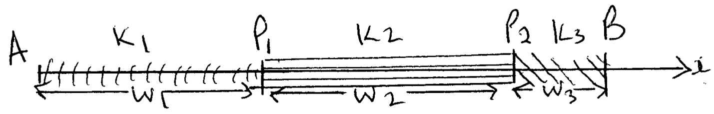
(a) Construct a condensed stiffness matrix and force vector for a linear element of width w for the problem. [10]

(b) Hence obtain an approximate solution to u at the end of each layer. [6]

(c) Given that the exact temperature at P_1, P_2 and B are $44.32, 32.97$ and 31.35 respectively, compare the results, with the aid if a graph. [3]

(d) Given that heat flow is given by $-K\frac{du}{dt}$, find heat flow in each element. [4]

(e) Compare the exact heat flow, 567.57 with the elements heat flow. [2]



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