NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 4202

## FACULTY OF APPLIED SCIENCES

### DEPARTMENT OF APPLIED MATHEMATICS

ADVANCED LINEAR ALGEBRA

NOV/DEC 2001



Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY FOUR** questions from Section B.

### SECTION A

**A1.** Compute $lim_{n\to\infty} A^n$ for $A = \begin{pmatrix} 7/5 & 1/5 \\ -1 & 1/2 \end{pmatrix}$ [8]

**A2.** Let $V$ and $W$ be vector spaces, and let $T : V \to W$ be a linear transformation. Prove that $T$ is one to one if and only if $ker(T) = \{0\}$ [4]

**A3.** If $k > 0$ is any integer and if $\mathbf{x}$ is an eigenvector corresponding to eigenvalue $\lambda$, show that $\mathbf{x}$ is also an eigenvector of $A^k$. Find the eigenvalue of $A^k$ corresponding to $\mathbf{x}$. If $A^k = I$, find all the possible real values of $\lambda$, distingushing between the two cases, $k$ even and $k$ odd. [6]

**A4.** Use the Gram-Schmidt procedure to find an orthonormal basis for the space spanned by the following linearly independent vectors:

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix}, x_3 = \begin{pmatrix} 3 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

[8]

**A5.** Draw a graph of the conic section

$$5x^2 - 6xy + 5y^2 = 8$$

[9]

**A6.** Prove that in a Euclidean vector space $(V, < >)$, the following identity holds:

$$||v + w||^2 + ||v - w||^2 = 2||v||^2 + 2||w||^2$$

What does this say about the diagonals of a parallelogram? [5]

## SECTION B

**B7.** (a) Explain what is meant by the saying that a transformation $T : V \to U$ between two vector spaces $V$ and $U$ is a linear transformation. Given that $S : V \to U$ is also a linear transformation, prove that the transformation $ST : V \to U$ defined by $ST(x) = S(T(x))$ and $S+T$ defined by $(S+T)(x) = S(x)+T(x)$ for all $x \in V$, are linear transformations.

(b) Prove that the following transformations are linear and calculate their null spaces (kernels)

(i) $T_1 : P \to P$ (where $P$ is the vectorspace of all real valued polynomials) defined by $T_1(p(x)) = xp(x)$ for all $p \in P$

(ii) $T_2 : \mathbf{R}^4 \to \mathbf{R}^4$ defined by

$$T_2\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_3 - x_4 \\ x_4 - x_1 \end{pmatrix}$$

[15]

**B8.** (a) Explain what is meant by an eigenvalue and eigenvector for a square matrix.

(b) Show that if $\mathbf{x}$ and $\mathbf{x}'$ are eigenvectors of a matrix $A$ corresponding to different eigenvalues $\lambda$ and $\lambda'$, then $\mathbf{x}$ and $\mathbf{x}'$ are linearly independent.

(c) Find the eigenvalues of the matrix

$$\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

and for each eigenvalue find the basis for its eigenspace. Hence write down the general solution of the system of differential equations:

$$\begin{aligned} \frac{dx}{dt} &= 4x + z \\ \frac{dy}{dt} &= -2x + y \\ \frac{dz}{dt} &= -2x + z \end{aligned}$$

[15]

**B9.** (a) Explain what is meant by an inner product on a real vector space $V$. Prove that in a real vector space with inner product *( a Euclidean space)* the following inequality holds for all $x, y \in V$:

$$| < x, y > |^2 \leq (< x, x >)(< y, y >)$$

(b) If $V$ is a Euclidean space and $f : V \to V$ is a mapping (not assumed linear) such that

$$* \quad < fx, fy > = < x, y >, \forall x, y \in V$$

Show that $f$ must be a linear transformation. **[HINT:** Let $u = f(\alpha x + \beta y) - \alpha(fx) - \beta(fy)$ and expand $< u, u >$ using *]

[15]

**B10.** (a) If possible, diagonalise the following matrix with a similarity transformation:

$$A = \begin{pmatrix} 1 & -4 & -4 \\ 8 & -11 & -8 \\ -8 & 8 & 5 \end{pmatrix}$$

(b) Find the canonical form of

$$\phi = 5x_1^2 + 6x_2^2 + 7x_3^2 - 4x_1x_2 + 4x_2x_3$$

and identify its type.

[7,8]

**B11.** (a) Show that if

$$A = \begin{bmatrix} 3/2 & 3/4 \\ -5/4 & -1/2 \end{bmatrix}$$

then

$$A^r = 1/2[-(3/4)^r + 3(1/4)^r]I + 2[(3/4)^r - (1/4)^r]A, r = 1, 2, \ldots,$$

Set

$$S_n = I + A + A^2 + \ldots + A^n = \sum_{r=0}^{n} A^r$$

where by definition $A^0 = I$. Then if as $n \to \infty$ the sum of the series forming the element in row $i$ and column $j$ of $S_n$ converges to the finite limit $S_{ij}$, infinite matrix series is said to have the sum $S$, where $s = [s_{ij}]$ and this is written as

$$S = \sum_{r=0}^{\infty} A^r$$

Use the matrix $A$ and the expression for $A^r$ in the first part of the problem to show that

$$\sum_{r=0}^{\infty} A^r = \frac{16}{3} A$$

(b) Let $T : P_1 \to P_2$ be the linear transformation defined by

$$T(p(x)) = xp(x)$$

Find the matrix for $T$ with respect to the standard bases

$$B = \{u_1, u_2\}, B' = \{v_1, v_2, v_3\}$$

where $u_1 = 1, u_2 = x$; and $v_1 = 1, v_2 = x, v_3 = x^2$

[15]

**B12.** Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear operator given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -2x^3 \\ x_1 + 2x_2 + x_3 \\ x_1 + 3x_3 \end{bmatrix}$$

Find a basis for $\mathbf{R}^3$ relative to which the matrix for $T$ is diagonal. [15]

**END OF QUESTION PAPER**