SMA4202

NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

SMA4202 ADVANCED LINEAR ALGEBRA

Nov/Dec 2002 Examination **Time 3 Hours**

Candidates should attempt **ALL** questions from Section **A** and **ANY FOUR** questions from Section B.

SECTION A

A1 Compute $\lim_{n\to\infty} A^n$ for

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$ **[8]**

A2 Find the canonical form of $\Phi = 5x_1^2 + 6x_2^2 + 7x_3^2 - 4x_1x_2 + 4x_2x_3$ and identify its type. **[8]**

A3 Let V and W be vector spaces, and let T: V $\rightarrow$ W be a linear transformation. Prove that T is one to one if and only if ker (T) = {0}. **[4]**

A4 Let (V, < >) be an inner product space. Let $\{e_1, e_2, ...\}$ be a sequence of vectors i.e elements of V. Say what is meant by $\{e_1, e_2, ...\}$ is orthonormal. Let $\{x_1, x_2, ...\}$ be linearly independent sequence of vectors. Write down the Gram – Schmidt formulae. for obtaining an orthonormal sequence $\{e_1, e_2, ...\}$ from $\{x_1, x_2, ...\}$. You need not show that $\{e_1, e_2, ...\}$ is orthonormal. **[10]**

A5 Let X be a vector space over **R**. Define what is meant by a norm on X.
Determine whether the following functions $\|\cdot\|$ are norms on the stated vector spaces X.

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(i) $X = \mathbf{R}, \|x\| = \frac{|x|}{1+|x|}, (X \in \mathbf{R})$

(ii) $X = \mathbf{R}, \|x\| = \frac{1}{2} |x| \ (x \in \mathbf{R})$ **[10]**

SECTION B

B6 Show that if $\mathbf{A} = \begin{bmatrix} \frac{3}{2} & \frac{3}{4} \\ \frac{-5}{4} & \frac{-1}{2} \end{bmatrix}$

then $A^r = \frac{1}{2}\left[-\left(\frac{3}{4}\right)^r + 3\left(\frac{1}{4}\right)^r\right] I + 2\left[\left(\frac{3}{4}\right)^r - \left(\frac{1}{4}\right)^r\right] A$, r = 1, 2, … where by definition $A^0 = I$. Then if as $n \to \infty$ the sum of the series forming the element in row i and column j of S_n converges to the finite limit S_{ij}, infinite matrix series is said to have the sum S, where $S = \lfloor s_{ij} \rfloor$ and this is written as

$$S = \sum_{r=0}^{\infty} A^r$$

Use the matrix A and the expression for A^r in the first part of the problem to show that

$$\sum_{r=0}^{\infty} A^r = \frac{16}{3} A$$ **[15]**

B7 (a) Define what is meant by the eigenvalues and eigenvectors of a matrix A.

(b) (i) If k> 0 is any integer and if x is an eigenvector corresponding to an eigenvalue λ.
Show that x is also an eigenvector of A^k

(ii) Find the eigenvalue of A^k corresponding to x. if $A^k = I$ find all possible real values of λ, distinguishing the two cases, k even and k odd.

(c) Find the eigenvalues of the matrix

$$A = \begin{bmatrix} -1 & 12 & -6 \\ 0 & -7 & 3 \\ 0 & -18 & 8 \end{bmatrix}$$

and for each eigenvalue find the basis for its eigenspace. Hence write down the general solution of the system of differential equations:

$$\frac{dx}{dt} = -x + 12y - 6z$$

$$\frac{dy}{dt} = -7y + 3z$$

$$\frac{dz}{dt} = -18y + 8z$$

[15]

B8 (a) Determine whether the matrix $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 3 \\ 1 & 3 & 1 \end{pmatrix}$ can be diagonalised by a similarity transformation.

(b) The vector space V has an inner product.

(i) State what it means to say that a linear transformation T: V $\rightarrow$ V is symmetric.

(ii) Show that if T is symmetric and λ is an eigenvalue of T then λ is real.

[15]

B9 Suppose T: V $\rightarrow$ W is a linear transformation. Define ker(T) and Im(T).
Prove that,
(i) ker T is a subspace of V
(ii) Im T is a subspace of W.

(b) Prove that the following transformations are linear transformations and calculate their kernels and their image spaces:

(i) $T_1 : P \rightarrow P$ (where P is the vector space of all real polynomials) defined by $T_1\,(p(x)) = \dfrac{d}{dx}\,(xp(x))$ for all $p \in P$.

(ii) $T_2 : R^4 \rightarrow R^4$ defined by

$$T_2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_3 - x_4 \\ x_4 - x_1 \end{pmatrix}$$

[15]

B10. Let $p(x) = c_0 + c_1x + \ldots\ldots\ldots + c_{nx}{}^k$ be a polynomial in P_n and define

$T : P_n \rightarrow P_{n+1}$ by $T(p(x)) = xP(x)$.

(a) Show that: (i) T is a linear transformation
(ii) T is one to one

(b) What is the ker(T) and Im(T)

(c) Find the matrix for T with respect to the standard bases

$B = \{u_1, u_2\}$, $B^1 = \{v_1, v_2, v_3\}$

where $u_1 = 1$, $u_2 = x$; *and* $v_1 = 1, v_2 = x, v_3 = x^2$ **[15]**

B11 Let T : $\mathbf{R}^3 \to \mathbf{R}^3$ by the linear operator given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} & & -2x_3 & \\ x_1 & + & 2x_2 & +x_3 \\ x_1 & + & 3x_3 & \end{bmatrix}$$

Find a basis for $\mathbf{R}^3$ relative to which the matrix for T is diagonal. [15]

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