NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF APPLIED MATHEMATICS

April/May 2001 EXAMINATION

SMA 4204 MATHEMATICAL METHODS

TIME: **3 HOURS**

ANSWER **ANY FOUR** QUESTIONS WITH AT LEAST **ONE** QUESTION FROM EACH SECTION

SECTION A:

1 Given is the boundary value problem

$$y''+\lambda y = 0, \quad 2y(0)+y'(0)=0, \quad y(1)=0.$$

a) Determine whether $\lambda = 0$ is an eigenvalue. [5]

b) Find the determinantal equation satisfied by the strictly positive eigenvalues. Show that there is an infinite sequence of such eigenvalues and that

$$\lambda_n = \frac{(2n+1)\pi}{2}$$

for large n. [12]

c) Find the determinantal equation satisfied by the negative eigenvalues and show that there is exactly one negative eigenvalue. [8]

2 a) Find the eigenfunction expansion $\sum_{n=1}^{\infty} a_n\phi_n(x)$ of $f(x)=x, \quad 0 \le x \le 1$, using the normalized eigenfunctions of the boundary value problem

$y''+\lambda y = 0, \quad y(0)=0, \quad y'(1)=0$. [7]

b) Solve the nonhomogeneous boundary value problem

$$y''+2y=-x, \quad y(0)=0, \quad y'(1)=0$$

by means of an eigenfunction expansion. [8]

c) Use an eigenfunction expansion to find the solution of the boundary value problem

$u_t = u_{xx} - x, \quad u(0,t)=0, \quad u_x(1,t)=0$

$u(x,0)=\sin\left(\frac{\pi x}{2}\right)$ [10]

SECTION B

3 a) Deduce the Euler equation relevant to the determination of an extremal for the integral

$$\int_0^1 \left[(y')^2 + k^2 \cos y\right] dx. \qquad [6]$$

b) Determine the stationary functions associated with the integral

$$\int_0^1 (y'^2 - 2ayy' - 2by')dx,$$

where a and b are constants, in each of the following situations:

(i) The end conditions $y(0) = 0$ and $y(1) = 1$ are preassigned. [7]

(ii) Only the end condition $y(0) = 0$ is preassigned. [4]

(iii) Only the condition $y(1) = 1$ is preassigned. [4]

(iv) No end conditions are preassigned. [4]

4 a) Find the geodesic on the surface of a right circular cone by using spherical coordinates with

$$ds^2 = dr^2 + r^2 \sin^2 \alpha d\theta^2. \qquad [10]$$

b) (i) Show that the extremals of the problem

$$\delta \int_{x_1}^{x_2} \left[p(x)y'^2 - q(x)y^2\right] dx = 0, \quad \int_{x_1}^{x_2} r(x)y^2 dx = 1,$$

where $y(x_1)$ and $y(x_2)$ are prescribed, are solutions of the equation

$$(py')' + (q + \lambda r)y = 0,$$

where λ is a constant. [9]

(ii) Show that the associated natural boundary conditions are of the form

$$\left[py'\delta y\right]_{x_1}^{x_2} = 0,$$

so that the same result follows if py' is required to vanish at an end point where y is not prescribed. [6]

SECTION C:

5 a) Reduce the initial value problem

$$\frac{d^2y}{dx^2} = \lambda y(x) + g(x), \quad y(0) = 1, \quad y'(1) = 0,$$

to a Volterra integral equation. [9]

b) (i) Show that, if $y(x)$ satisfies the initial value problem

$$y'' + xy = 1, \quad y(0) = y'(0) = 0,$$

then $y(x)$ also satisfies the Volterra equation

$$y(x) = \int_0^x \xi(\xi - x)y(\xi)d\xi + \frac{1}{2}x^2.$$

(ii) Prove that the reverse to the statement in (i) is true. [9;7]

6 a) Transform the problem

$$\frac{d^2y}{dx^2} + y = x, \quad y(0) = 1, \quad y'(1) = 0$$

to a Fredholm integral equation. [8]

b) Verify that the Green's function for the Bessel operator of order n,

$$L[y] = (xy')' - \frac{n^2}{x}y,$$

relevant to the end conditions $y(0) = y(1) = 0$, is of the form

$$G(x,\xi) = \begin{cases} \dfrac{x^n/\xi^n}{2n}(1 - \xi^{2n}), & x < \xi \\ \dfrac{\xi^n/x^n}{2n}(1 - x^{2n}), & x > \xi, \end{cases}$$

when $n \neq 0$. Hence reduce the problem

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda x^2 - n^2)y = 0, \quad y(0) = y(1) = 0$$

to an integral equation, when $n \neq 0$. [17]