

NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF APPLIED MATHEMATICS

JULY 2001 SUPPLEMENTARY EXAMINATION

SMA 4204 MATHEMATICAL METHODS

TIME:

3 HOURS

ANSWER ANY FOUR QUESTIONS WITH AT LEAST ONE QUESTION FROM EACH SECTION

* LIBRARY USE ONLY *

SECTION A:

- 1 Given is the boundary value problem
 $y'' + \lambda y = 0, \quad y(0) + y'(0) = 0, \quad y(1) = 0.$
- a) Determine whether $\lambda = 0$ is an eigenvalue. [5]
- b) Find the determinantal equation satisfied by the nonzero eigenvalues. Show that there is an infinite sequence of such eigenvalues and that
$$\lambda_n = -\frac{(2n+1)^2 \pi^2}{4}$$
for large n. [12]
- c) Find an approximate value for λ_1 , the nonzero eigenvalue of smallest absolute value. [8]
- 2 a) Find the eigenfunction expansion $\sum_{n=1}^{\infty} a_n \phi_n(x)$ of $f(x) = x, \quad 0 \leq x \leq 1$, using the normalized eigenfunctions of the boundary value problem
 $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(1) = 0.$ [10]
- c) Use an eigenfunction expansion to find the solution of the boundary value problem
 $u_t = u_{xx} - x, \quad u(0,t) = 0, \quad u_x(1,t) = 0$
 $u(x,0) = \sin\left(\frac{\pi x}{2}\right)$ [15]

SECTION B :

- 3 a) Deduce the Euler equation relevant to the determination of an extremal for the integral

$$\int_0^1 (1 + y'^2)^{1/2} dx. \quad [6]$$

- b) Determine the stationary functions associated with the integral

$$\int_a^b (xy'^2 - yy' + y) dx,$$

where a and b are constants, in each of the following situations:

- (i) The end conditions $y(0) = 0$ and $y(1) = 1$ are preassigned. [7]
 (ii) Only the end condition $y(0) = 0$ is preassigned. [4]
 (iii) Only the condition $y(1) = 1$ is preassigned. [4]
 (iv) No end conditions are preassigned. [4]

- 4 a) Find the geodesic on the surface of a right circular cone by using spherical coordinates with

$$ds^2 = dr^2 + r^2 \sin^2 \alpha d\theta^2. \quad [10]$$

- b) (i) Show that the extremals of the problem

$$\delta \int_a^b [p(x)y'^2 - q(x)y^2] dx = 0, \quad \int_a^b r(x)y^2 dx = 1,$$

where $y(x_1)$ and $y(x_2)$ are prescribed, are solutions of the equation

$$(py') + (q + \lambda r)y = 0,$$

where λ is a constant. [9]

- (ii) Show that the associated natural boundary conditions are of the form

$$[py' \delta y]_{x_1}^{x_2} = 0,$$

so that the same result follows if py' is required to vanish at an end point where y is not prescribed. [6]

SECTION C:

5 Consider the initial value problem

$$\begin{cases} \frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y(x) = f(x) \\ y(a) = c_1, \quad y'(a) = c_2. \end{cases}$$

Show that this problem can be represented in the form of a Volterra integral equation of the second kind. [25]

6 a) Transform the problem

$$\frac{d^2y}{dx^2} + y = x, \quad y(0) = 1, \quad y'(1) = 0$$

to a Fredholm integral equation. [8]

b) Verify that the Green's function for the Bessel operator of order n ,

$$L[y] = (xy')' - \frac{n^2}{x}y,$$

relevant to the end conditions $y(0) = y(1) = 0$, is of the form

$$G(x, \xi) = \begin{cases} \frac{x^n / \xi^n}{2n} (1 - \xi^{2n}), & x < \xi \\ \frac{\xi^n / x^n}{2n} (1 - x^{2n}), & x > \xi, \end{cases}$$

when $n \neq 0$. Hence reduce the problem

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (\lambda x^2 - n^2)y = 0, \quad y(0) = y(1) = 0$$

to an integral equation, when $n \neq 0$. [17]