National UniversityOf Science and Technology

Department of Applied Mathematics

May 2002 Examination

SMA 4204 Mathematical Methods

Time: **3 Hours**

ANSWER **ANY FOUR** QUESTIONS WITH AT LEAST **ONE** QUESTION FROM EACH SECTION

SECTION A:

1 a) Consider the heat flow with convection problem

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - V_0\frac{\partial u}{\partial x},$$

where k and V_0 are constants. Show that the spatial ordinary differential equation obtained by separation of variables is not in Sturm-Liouvile form. **[4]**

b) For the eigenvalue problem

$$\frac{d^2y}{dx^2} + \lambda y = 0, \quad y(0) = 0 \text{ and } y(\pi) - 2\frac{dy(0)}{dx} = 0 ,$$

i. Determine all positive eigenvalues. **[6]**

ii. Is $\lambda = 0$ an eigenvalue? **[3]**

c) For the problem

$$y''+\lambda y = 0, \quad y(0) = 0 \text{ and } y'(1) = 0,$$

show that if ϕ_m and ϕ_n are eigenfunctions corresponding to the eigenvalues λ_m and λ_n respectively, with $\lambda_m \neq \lambda_n$, then

$$\int_0^1 \phi_m(x)\phi_n(x)dx = 0.$$ **[4]**

d) Find the eigenfunction expansion of the function

$$f(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 1, & \frac{1}{2} \le x \le 1 \end{cases}$$

using the normalized eigenfunctions of Problem 1c) above. **[8]**

2 a) Solve the nonhomogeneous problem by the method of an eigenfunction expansion

$y''+2y = -1 + |1-2x|$, $y(0) = 0$, $y(1) = 0$. **[10]**

b) Consider the nonhomogeneous initial-boundary value problem

$$u_t = u_{xx} + 1 - |1-2x|, \; u(0,t) = 0, \; u(1,t) = 0, \; u(x,0) = 0.$$

Use the eigenfunction expansion method to solve this problem. **[15]**

SECTION B :

3 a) Determine the Euler equation relevant to the determination of an extremal for the integral $\int_0^{1}(xy'^2 - yy' + y)dx$. **[6]**

b) Determine the stationary function associated with the integral

$$\int_0^{1}(y'^2)f(x)dx,$$

and the conditions $y(0) = 0$ and $y(1) = 1$, where

$$f(x) = \begin{cases} -1 , & 0 \le x < \frac{1}{4} \\ 1 , & \frac{1}{4} < x \le 1 \end{cases}.$$ **[6]**

c) If l is not preassigned, show that the stationary functions corresponding to the problem

$$\begin{cases} \delta \int_0^{l}\{y'^2 + 4(y - l)\}dx = 0 , \\ y(0) = 2 \ , \ y(l) = l^2 \end{cases}$$

are of the form $y = x^2 - 2\frac{x}{l} + 2$, where l is one of the two real roots of the equation $2l^4 - 2l^3 - 1 = 0$. **[13]**

4 a) Consider the problem

$$\delta\left\{\int_a^b F(x, y, y')dx - \beta y(b) + \alpha y(a)\right\} = 0 ,$$

where α and β are given constants but $y(a)$ and $y(b)$ are not preassigned. Determine the Euler's equation and the associated natural boundary conditions for this problem. **[7]**

b)(i) Consider the problem

$$\delta \int_a^b \{s(x)(y'')^2 - p(x)y'^2 + q(x)y^2\}dx = 0, \quad \int_a^b r(x)y^2\,dx = 1,$$

where the values of the functions $y(x)$, $y'(x)$ are prescribed at the end points. Determine the Euler's equation representing the extremals of this problem. **[9]**

(ii)By using the associated natural boundary conditions, show that the same result, as above in (i), follows if $(sy'')' + py'$ is set equal to zero at an end point where y is not given, and sy'' is required to vanish at an end point where y' is not given. **[3]**

(iii)Hence determine the extremals of the problem

$$\delta \int_0^{\pi} y''^2\,dx = 0, \quad \int_0^{\pi} y^2\,dx = 0$$
$$y(0) = y''(0) = 0, \ y(\pi) = y''(\pi) = 0.$$ **[6]**

SECTION C

5 Consider the boundary value problem

$$y'' = F(x)$$

$$y(0) = 0, \quad y(1) = 0.$$

(i) Show that

$$y(x) = \int_0^x (x-t)F(t)dt - x\int_0^1 (1-t)F(t)dt.$$ **[6]**

(ii) a) Show that the integral form of the solution $y(x)$ given in (i) above can be written in the form

$$y(x) = \int_0^1 K(x,t)F(t)dt$$ **[6]**

b) Define $K(x,t)$. **[2]**

c) What do we call the integral equation in (ii) a)? **[1]**

(iii) Verify directly that the integral in (ii) a) satisfies the given differential equation and the boundary conditions. **[3]**

b) For the initial value problem

$$y'' + xy = 1$$

$$y(0) = y'(0) = 0$$

(i) show that, if $y(x)$ satisfies the initial value problem, then y also satisfies the equation

$$y(x) = \int_0^x t(t-x)y(t)dt + \frac{x^2}{2}.$$ **[6]**

What do we call this integral equation? **[1]**

6 Consider the boundary value problem

$$x^2y''(x) + xy'(x) + (\lambda x^2 - 1)y(x) = 0 \qquad (i)$$

$$y(0) = y(1) = 0 \qquad (ii)$$

a) Determine the function $p(x)$ such that

$$p(x)y''(x) + p(x)\frac{1}{x}y'(x) = \frac{d}{dx}\left[p(x)y'(x)\right].$$

Hence write (i) in standard form. [7]

b) Obtain the equivalent integral equation, $y(x) = \lambda\int_0^1 G(x,t)ty(t)dt$, where $G(x,t)$ is the Green's function, representation of the boundary value problem $(i),(ii)$. **[18]**