NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF APPLIED MATHEMATICS

~~JULY~~ May 2003 EXAMINATION

SMA 4204 MATHEMATICAL METHODS

TIME: **3 HOURS**

ANSWER **ANY FOUR** QUESTIONS WITH AT LEAST **ONE** QUESTION FROM EACH SECTION

SECTION A:

1 Given the boundary value problem

$$y''+\lambda y = 0, \quad y(0)+y'(0)=0, \quad y(1)=0,$$

(i) Determine whether $\lambda = 0$ is an eigenvalue. **[5]**

(ii) Find the determinantal equation satisfied by the nonzero eigenvalues. Show that there is an infinite sequence of such eigenvalues and that

$$\lambda_n = \frac{(2n+1)^2\pi^2}{4}$$

for large n. **[12]**

(iii) Find an approximate value for λ_1, the nonzero eigenvalue of smallest absolute value. **[8]**

2. (i) Consider the boundary value problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(1) + y'(1) = 0.$$

The nonzero eigenvalues λ_n of this problem satisfy the determinantal equation

$$\sqrt{\lambda_n} = \cot\sqrt{\lambda_n}$$

and the corresponding eigenfunctions are $\phi_n(x) = k_n \cos\sqrt{\lambda_n}x$ where k_n is an arbitrary constant.

a) Determine the normalized eigenfunctions of the boundary value problem. **[6]**

b) Expand the function

$$f(x) = x, \quad 0 \le x \le 1$$

in terms of the normalized eigenfunctions $\phi_n(x)$ of the boundary value problem. **[12]**

(ii) Determine whether the boundary value problem

$$y'' + y' + 2y = 0, \quad y(0) = 0, \quad y(1) = 0$$

is self-adjoint. **[7]**

3 (i) Solve the following nonhomogeneous problem by the method of eigenfunction expansion

$$y''+2y = -1+|1-2x|, \ y(0) = 0, \ y(1) = 0.$$ **[8]**

(ii) Consider the nonhomogeneous initial-boundary value problem

$$u_t = u_{xx} + 1 - |1-2x|, \ u(0,t) = 0, \ u(1,t) = 0, \ u(x,0) = 0.$$

Use the eigenfunction expansion method to solve this problem. **[17]**

SECTION B :

4 (i) Let $y = 1 + x^2$, where x and y are functions of an independent variable t.

a) Calculate $\delta \dfrac{dy}{dx}$ and $\dfrac{d}{dx}\delta y$ where $x = \dfrac{1}{t}$ and $\delta x = \varepsilon t^2$. **[7]**

b) Verify the validity of the equation

$$\delta \frac{dy}{dx} = \frac{d}{dx}\delta y - \frac{dy}{dx}\frac{d}{dx}\delta x$$

in this case. **[5]**

(ii) If l is not preassigned, show that the extremals corresponding to the problem

$$\delta \int y'^2 dx = 0; \quad y(0) = 2, \quad y(l) = \sin l$$

are of the form $y = 2 + 2x\cos l$, where l satisfies the transcendental equation $2 + 2l\cos l - \sin l = 0$. **[13]**

5 (i) Find the geodesic on the surface of a right circular cone by using spherical coordinates with $ds^2 = dr^2 + r^2 \sin^2 \alpha d\theta^2$. **[15]**

(ii) a) Show that the extremals of the problem

$$\delta \int_{x_1}^{x_2} \left[p(x)y'^2 - q(x)y^2\right] dx = 0, \quad \int_{x_1}^{x_2} r(x)y^2 dx = 1,$$

where $y(x_1)$ and $y(x_2)$ are prescribed, are solutions of the equation

$$(py')' + (q + \lambda r)y = 0,$$

where λ is a constant. **[7]**

b) Show that the associated natural boundary conditions are of the form

$$\left[py'\delta y\right]_{x_1}^{x_2} = 0,$$

so that the same result follows if py' is required to vanish at an end point where y is not prescribed. **[3]**

SECTION C:

6 (i) Consider the boundary value problem

$$y'' = F(x) \quad ; \quad y(0) = 0, \quad y(1) = 0.$$

a) Show that

$$y(x) = \int_0^x (x-t)F(t)dt - x\int_0^1 (1-t)F(t)dt.$$ **[9]**

b) Show that the integral form of the solution $y(x)$ given in (i) above can be written in the form

$$y(x) = \int_0^1 K(x,t)F(t)dt$$

where $K(x,t)$ is a kernel. **[6]**

What do we call the integral equation obtained? **[1]**

(ii) For the initial value problem

$$y'' + xy = 1$$
$$y(0) = y'(0) = 0$$

show that, if $y(x)$ satisfies the initial value problem, then y also satisfies the equation

$$y(x) = \int_0^x t(t-x)y(t)dt + \frac{x^2}{2}.$$ **[9]**

What do we call this integral equation? **[1]**

7 (i) Determine $p(x)$ and $q(x)$ in such a way that the equation

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0,$$

is equivalent to the equation

$$L[y] = \frac{d}{dx}\left(p\frac{dy}{dx}\right)qy = 0.$$ **[8]**

(ii) a) Derive Abel's formula given that $u = u(x)$ and $v = v(x)$ are functions of x satisfying the equation $L[y] = 0$ in their intervals of definition. **[4]**

b) Verify that $u = x$ and $v = x^2$ are linearly independent solutions of the equation of part (i). What is the value of the constant A in the Abel's formula for our case? **[4:2]**

(iii) Show that, if we take $u(x) = a_1x + a_2x^2$ and $v(x) = b_1x + b_2x^2$, then $A = a_1b_2 - a_2b_1$. **[7]**