NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 4204

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

MATHEMATICAL METHODS

June 2004

Time : 3 hours

Candidates Should Answer **ANY FOUR** Questions With **AT LEAST ONE** From Each Section

SECTION A:

A1. Given the boundary value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(1) = 0$$

Determine the determinantal equation satisfied by non-zero eigenvalues of the boundary value problem (BVP) and hence find the normalised eigenfunctions of the BVP. [6]

(b) Expand the function

$$f(x) = \begin{cases} 2x; & 0 \leq x < \frac{1}{2} \\ 1; & \frac{1}{2} \leq x < 1 \end{cases}$$

in terms of normalised eigenfunction of the BVP. [12]

(c) Determine whether the BVP

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) + y'(\pi) = 0$$

is self-adjoint. [7]

A2. (a) Consider the problem

$$r(x)u_{tt} = [p(x)u_x]_x - q(x)u + F(x,t) \qquad (i)$$

$$u_x(0,t) - h_1 u(0,t) = 0, \qquad u_x(1,t) + h_1 u(1,t) \qquad (ii)$$

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x) \qquad (iii)$$

This problem can arise in connection with generalisations of the telegraph equation or longitudinal vibrations of an elastic bar.

(i) Let $u(x,t) = X(x)T(t)$ in the homogeneous equation corresponding to equation (i). Then deduce the BVP associated with the homogeneous differential equation and the boundary conditions (ii). Let λ_n and $\varphi_n(x)$ be the eigenvalues and the normalised eigenfunctions of the derived BVP.

(ii) Suppose that $u(x,t)$ can be expressed as a series of eigenfunctions,

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\varphi_n(x)$$

Show that $b_n(t)$ satisfies the initial value problem

$$b_n''(t) + \lambda_n b_n(t) = \gamma_n(t), \qquad b_n(0) = \alpha_n, \qquad b_n'(0) = \beta_n$$

where α_n, β_n, $\gamma_n(t)$ are expansion coefficients for $f(x)$, $g(x)$ and $\frac{F(x,t)}{r(x)}$ respectively in terms of the eigenfunctions $\varphi_1(x)$, $\varphi_2(x)$,.......$\varphi_n(x)$,....

(b) Use the eigenfunction expansion to find the solution of the boundary value problem

$$u_t = u_{tt} + e^{-t}; \quad u_x(0,t) = 0, \quad u_x(1,t) + u(1,t) = 0, \quad u(x,0) = 1 - x$$

where $\varphi_n(x) = \left(\frac{2}{1+sin^2\sqrt{\lambda_n}}\right)^{\frac{1}{2}} \cos\sqrt{\lambda_n}x$ are the normalised eigenfunctions of the corresponding homogeneous problem and λ_n satisfies the equation

$$\cos\sqrt{\lambda_n} - \sqrt{\lambda_n}\sin\sqrt{\lambda_n} = 0$$

[25]

SECTION B:

B3. (a) The Brachistochrome "shortest-time" problem posed by Bernoulli in 1696, states that a particle of mass m starts at a point $P_1(x_1, y_1)$ with speed V and moves under gravity (acting in the negative y direction) along curve $y = f(x)$ to a point $P_2(x_2, y_2)$ where $y_1 > y_2$ and $x_2 > x_1$, and requires the curve along which the elapsed time T is minimum. If v is the speed and s the distance travelled at time t, use the relation $\frac{ds}{dt} = v$ and $\frac{1}{2}mv^2 + mgy = constant$ to obtain the formulation:

$$T = \frac{1}{\sqrt{2g}}\int_0^T \frac{\sqrt{1+y'^2}}{\sqrt{\alpha - y}}dx = minimum, \qquad \left(\alpha = y_1 + \frac{V^2}{2g}\right)$$

where $y(x_1) = y_1$ and $y(x_2) = y_2$. [9]

(b) Determine the stationary function associated with the integral

$$\int_0^1 (y'^2 - 2ayy' - 2by')dx$$

where a and b are constants, in each of the following situations.

(i) The end conditions $y(0) = 0$ and $y(1) = 1$ are preassigned. [4]

(ii) Only the end condition $y(0) = 0$ is preassigned. [4]

(iii) Only the condition $y(1) = 1$ is preassigned. [4]

(iv) No end conditions are preassigned. [4]

B4. (a) Derive the Euler equation of the problem

$$\delta \int_{x_1}^{x_2} F(x, y, y', y'') = 0$$

in the form

$$\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial F}{\partial y} = 0$$

and show that the associated natural boundary conditions (NBCs) are

$$\left[\left(\frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right) - \frac{\partial F}{\partial y'}\right)\delta y\right]_{x_1}^{x_2} = 0 \quad \text{and} \quad \left[\frac{\partial F}{\partial y''}\delta y'\right]_{x_1}^{x_2}$$

[10]

(b) (i) Show that the extremals of the problem

$$\delta \int_{x_1}^{x_2} \left[p(x)y'^2 - q(x)y^2\right] dx = 0, \qquad \int_{x_1}^{x_2} r(x)y^2 dx = 1$$

where $y(x_1)$ and $y(x_2)$ are prescribed, are solutions of the equation:

$$[py']' + (q - \lambda r)y = 0$$

[9]

(ii) Show that the associated NBCs are of the form $[py'\delta y]_{x_1}^{x_2} = 0$ so that the same result follows if py' is required to vanish at the end point where y is not prescribed. [6]

SECTION C:

C5. Consider the boundary value problem

$$\frac{d^2y}{dx^2} = \lambda y(x), \qquad a < x < b \qquad (i)$$

$$y(a) = 0 \qquad , \qquad y(b) = 0 \qquad (ii)$$

Reduce the BVP (i),(ii) into a Fredholm Integral Equation. [25]

C6. Consider the boundary value problem

$$p(x)y''(x) + xy'(x) + (\lambda x^2 - 1)y(x) = 0 \qquad (i)$$

$$y(0) = y(1) = 0 \qquad (ii)$$

(a) Determine the function $p(x)$ such that

$$p(x)y''(x) + p(x)\frac{1}{x}y'(x) = \frac{d}{dx}\left[p(x)y'(x)\right]$$

Hence write (i) in standard form. [5]

(b) Obtain the equivalent integral equation,

$$y(x) = \lambda \int_0^1 G(x,t)ty(t)dt$$

where $G(x,t)$ is the Green's function representation of the BVP (i),(ii). [20]

END OF QUESTION PAPER