# NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 4204

## FACULTY OF APPLIED SCIENCES

## DEPARTMENT OF APPLIED MATHEMATICS

MATHEMATICAL METHODS: SUPPLEMENTARY EXAMINATION

August 2004

Time : 3 hours

Candidates Should Answer **ANY FOUR** Questions With **AT LEAST ONE** From Each Section

## SECTION A:

**A1.** Given the boundary value problem

$$y'' + \lambda y = 0, \quad 2y(0) + y'(0) = 0, \quad y(1) = 0$$

(a) Determine whether $\lambda = 0$ is an eigenvalue. [5]

(b) Find the determinantal equation satisfied by the strictly positive eigenvalues. Show that there is an infinite sequence of such eigenvalues and that

$$\lambda_n = \frac{(2n+1)^2\pi^2}{4} \qquad \text{for large n.}$$

[12]

(c) Find the determinantal equation satisfied by the negative eigenvalues and show that there is exactly one negative eigenvalue. [8]

**A2.** (a) Consider the problem

$$r(x)u_{tt} = [p(x)u_x]_x - q(x)u + F(x,t) \qquad (i)$$

$$u_x(0,t) - h_1 u(0,t) = 0, \qquad u_x(1,t) + h_1 u(1,t) \qquad (ii)$$

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x) \qquad (iii)$$

This problem can arise in connection with generalisations of the telegraph equation or longitudinal vibrations of an elastic bar.

(i) Let $u(x,t) = X(x)T(t)$ in the homogeneous equation corresponding to equation $(i)$. Then deduce the BVP associated with the homogeneous differential equation and the boundary conditions $(ii)$. Let $\lambda_n$ and $\varphi_n(x)$ be the eigenvalues and the normalised eigenfunctions of the derived BVP.

(ii) Suppose that $u(x,t)$ can be expressed as a series of eigenfunctions,

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\varphi_n(x)$$

Show that $b_n(t)$ satisfies the initial value problem

$$b_n''(t) + \lambda_n b_n(t) = \gamma_n(t), \qquad b_n(0) = \alpha_n, \qquad b_n'(0) = \beta_n$$

where $\alpha_n$, $\beta_n$, $\gamma_n(t)$ are expansion coefficients for $f(x)$, $g(x)$ and $\frac{F(x,t)}{r(x)}$ respectively in terms of the eigenfunctions $\varphi_1(x)$, $\varphi_2(x)$,.......$\varphi_n(x)$,....

(b) Use the eigenfunction expansion to find the solution of the boundary value problem

$$u_t = u_{xx} + e^{-t}; \quad u_x(0,t) = 0, \quad u_x(1,t) + u(1,t) = 0, \quad u(x,0) = 1 - x$$

where $\varphi_n(x) = \left(\frac{2}{1+sin^2\sqrt{\lambda_n}}\right)^{\frac{1}{2}} \cos\sqrt{\lambda_n}x$ are the normalised eigenfunctions of the corresponding homogeneous problem and $\lambda_n$ satisfies the equation

$$\cos\sqrt{\lambda_n} - \sqrt{\lambda_n}\sin\sqrt{\lambda_n} = 0$$

[25]

---

## SECTION B:

**B3.** (a) Deduce the Euler equation relevant to the determination of an extremal for the integral

$$\int_0^1 \left[(y')^2 + k^2\cos y\right] dx$$

[6]

(b) Determine the stationary function associated with the integral

$$\int_0^1 (y'^2 - 2ayy' - 2by')dx$$

where $a$ and $b$ are constants, in each of the following situations.

(i) The end conditions $y(0) = 0$ and $y(1) = 1$ are preassigned. [7]

(ii) Only the end condition $y(0) = 0$ is preassigned. [4]

(iii) Only the condition $y(1) = 1$ is preassigned. [4]

(iv) No end conditions are preassigned. [4]

**B4.** (a) Find the geodesic on the surface of a right circular cone by using spherical coordinates with

$$ds^2 = dr^2 + r^2 \sin^2 \alpha d\theta^2$$

[10]

(b) (i) Show that the extremals of the problem

$$\delta \int_{x_1}^{x_2} \left[p(x)y'^2 - q(x)y^2\right] dx = 0, \qquad \int_{x_1}^{x_2} r(x)y^2 dx = 1$$

where $y(x_1)$ and $y(x_2)$ are prescribed, are solutions of the equation:

$$[py']' + (q - \lambda r)y = 0$$

[9]

(ii) Show that the associated NBCs are of the form $[py'\delta y]_{x_1}^{x_2} = 0$ so that the same result follows if $py'$ is required to vanish at the end point where $y$ is not prescribed. [6]

## SECTION C:

**C5.** Consider the initial value problem

$$\frac{d^2y}{dx^2} = \lambda y(x) + g(x), \qquad (i)$$

$$y(0) = 1 \quad , \quad y'(0) = 0 \qquad (ii)$$

(a) Reduce the IVP (i),(ii) into a Voltera Integral Equation. [9]

(b) (i) Show that, if $y(x)$ satisfies the initial value problem

$$y'' + xy = 1, \quad y(0) = y'(0) = 0$$

then $y(x)$ also satisfies the Voltera equation

$$y(x) = \int_0^x \zeta(\zeta - x)y(\zeta)d\zeta + \frac{1}{2}x^2$$

[9]

(ii) Prove that the reverse to the statement in (i) is true. [7]

**C6.** Consider the boundary value problem

$$p(x)y''(x) + xy'(x) + (\lambda x^2 - 1)y(x) = 0 \qquad (i)$$

$$y(0) = y(1) = 0 \qquad (ii)$$

(a) Determine the function $p(x)$ such that

$$p(x)y''(x) + p(x)\frac{1}{x}y'(x) = \frac{d}{dx}\left[p(x)y'(x)\right]$$

Hence write (i) in standard form. [5]

(b) Obtain the equivalent integral equation,

$$y(x) = \lambda \int_0^1 G(x,t)ty(t)dt$$

where $G(x,t)$ is the Green's function representation of the BVP (i),(ii). [20]

**END OF QUESTION PAPER**