NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 4204

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

SMA4204: MATHEMATICAL METHODS

MAY 2005

Time : 3 hours

Candidates should attempt **FOUR** questions with **AT LEAST ONE** from each section

SECTION A:

A1. Consider the BVP

$$-(xy')' = \lambda xy \qquad (i)$$

$$y, y' \text{are bounded as } x \to 0 \qquad (ii)$$

$$y'(1) = 0 \qquad (iii)$$

(a) Show that $\lambda_0 = 0$ is an eigenvalue of this problem corresponding to the eigenfunction $\phi_0(x) = 1$. If $\lambda > 0$, show formally that the eigenfunctions are given by $\phi_n(x) = J_0(\sqrt{\lambda}x)$ where $\sqrt{\lambda_n}$ is the n^{th} positive root of the equation

$$J_0'(\sqrt{\lambda}) = 0, \text{ where } J_0(\sqrt{\lambda}x) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m \lambda^m x^{2m}}{2^{2m}(m!)^2}, \quad x > 0$$

and

$$Y_0(\sqrt{\lambda}x) = \frac{2}{\pi}\left\{\left(\gamma + \ln\frac{\sqrt{\lambda}x}{2}\right) J_0(\sqrt{\lambda}x) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m \lambda^m x^{2m}}{2^{2m}(m!)^2}\right\}, \quad x > 0$$

where

$$H_m = 1 + \frac{1}{2} + \ldots\ldots\ldots + \frac{1}{m} \quad \text{and } \gamma = \lim_{m\to\infty}(H_m - \ln m).$$

(b) Show that the singular BVP for equation (i) is self-adjoint. [25]

A2. (a) Consider the boundary value problem

$$y'' + \lambda y = 0, \; y'(0) = 0, \; y(1) + y'(1) = 0.$$

The nonzero eigenvalues λ_n of this problem satisfy the determinantal equation

$$\sqrt{\lambda_n} = \cot \sqrt{\lambda_n}$$

and the corresponding eigenfunctions are $\phi_n(x) = k_n \cos \sqrt{\lambda_n} x$ where k_n is an arbitrary constant.

(i) Determine the normalised eigenfunctions of the boundary value problem. [6]

(ii) Expand the function

$$f(x) = x, \; 0 \leq x \leq 1$$

in terms of the normalised eigenfunctions $\phi_n(x)$ of the boundary value problem. [12]

(b) Determine whether the boundary value problem

$$y'' + y' + 2y = 0, \; y(0) = y(1) = 0$$

is self-adjoint. [7]

SECTION B:

B3. It is required to find an extremal of the functional

$$\int_a^b F(x, y(x), y'(x), y''(x))dx$$

among all smooth functions $y(x)$ which satisfy the boundary conditions $y(a) = y(b) = 0$.

(a) Show that such an extremal must be a solution of the differential equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$$

and must satisfy the Natural Boundary Conditions

$$\frac{\partial F}{\partial y''} = 0 \text{ at } x = a \text{ and } x = b.$$

[10]

(b) An elastic beam has vertical displacement $y(x), x \in [0, l]$. (The x-axis is horizontal and the y-axis is vertical and directed upwards). The ends of the beam are supported, that is $y(0) = y(l) = 0$, and the displacement minimises the energy

$$\int_0^l \left\{ \frac{1}{2} D \left[y''\right]^2 + \rho g y(x) \right\} dx$$

where D, ρ and g are positive constants. Write down the differential equation and the boundary conditions that $y(x)$ must satisfy and show that

$$y(x) = -\frac{\rho g}{24D} x(l-x)(l^2 + x(l-x)).$$

[15]

B4. (a) Two points $A = (a, y_0)$, $B = (b, y_1)$ in the region $y > 0$ with $b > a$ are to be joined by a continuously differentiable plane curve C lying in the same region. The curve C is to be chosen to minimise

$$I = \int_C \frac{ds}{y}$$

where ds is the element of arc length along the curve C from A to B.
Show that the minimum value of I occurs when C is a circular arc with its centre on the x-axis. [8]

(b) Find the extremal corresponding to

$$\int_{-1}^{1} y dx \quad \text{when subject to} \quad y(-1) = y(1) = 0, \quad \text{and}$$

$$\int_{-1}^{1} (y^2 + y'^2) dx = 1.$$

[7]

(c) Determine the equation of a geodesic on a right circular cylinder. (*Hint:* $ds^2 = r^2 d\theta^2 + dz^2, r = constant$) [10]

SECTION C:

C5. (a) Transform the problem

$$\frac{d^2y}{dx^2} + y = x, \; y(0) = 1, \; y'(1) = 0, \text{ to a Fredholm Integral Equation.}$$

[8]

(b) Verify that the Green's function for the Bessel operator of order n,

$$L[y] = (xy')' - \frac{n^2}{x}y,$$

relevant to the end conditions $y(0) = y(1) = 0$ is of the form

$$G(x, \zeta) = \begin{cases} \frac{x^n/\zeta^n}{2n}(1 - \zeta^{2n}), & x < \zeta \\ \frac{\zeta^n/x^n}{2n}(1 - x^{2n}), & x > \zeta \end{cases}$$

when $n \neq 0$. Hence reduce the problem

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda x^2 - n^2)y = 0, \; y(0) = y(1) = 0$$

to an integral equation, when $n \neq 0$. [17]

C6. (i) Consider the boundary value problem

$$y'' = F(x), \; y(0) = y(1) = 0.$$

(a) Show that

$$y(x) = \int_0^x (x - t)F(t)dt - x\int_0^1 (1 - t)F(t)dt.$$

[9]

(b) (i) Show that the integral form of the solution $y(x)$ given above can be written in the form

$$y(x) = \int_0^1 K(x, t)F(t)dt$$

where $K(x, t)$ is a kernel. [6]

(ii) What do we call the integral equation obtained? [1]

(ii) (a) For the initial value problem

$$y'' + xy = 1, \; y(0) = y'(0) = 0$$

show that, if $y(x)$ satisfies the initial value problem, then y also satisfies the equation

$$y(x) = \int_0^x t(t - x)y(t)dt + \frac{x^2}{2}.$$

[8]

(b) What do we call this integral equation? [1]