NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF APPLIED MATHEMATICS

SMA4204 MATHEMATICAL METHODS

July 2005

Time: 3 Hours

Candidates should attempt **FOUR** questions with at **LEAST ONE** question from each section

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SECTION A:

1 Given is the boundary value problem

$$y''+\lambda y = 0, \quad 2y(0)+y'(0) = 0, \quad y(1) = 0.$$

a) Determine whether $\lambda = 0$ is an eigenvalue. [5]

b) Find the determinantal equation satisfied by the strictly positive eigenvalues. Show that there is an infinite sequence of such eigenvalues and that

$$\lambda_n = \frac{(2n+1)\pi}{2}$$

for large n. [12]

c) Find the determinantal equation satisfied by the negative eigenvalues and show that there is exactly one negative eigenvalue. [8]

2. (i) Consider the boundary value problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(1) + y'(1) = 0.$$

The nonzero eigenvalues λ_n of this problem satisfy the determinantal equation

$$\sqrt{\lambda_n} = \cot\sqrt{\lambda_n}$$

and the corresponding eigenfunctions are $\phi_n(x) = k_n \cos\sqrt{\lambda_n}x$ where k_n is an arbitrary constant.

a) Determine the normalized eigenfunctions of the boundary value problem. [6]

b) Expand the function

$$f(x) = x, \quad 0 \le x \le 1$$

in terms of the normalized eigenfunctions $\phi_n(x)$ of the boundary value problem. [12]

(ii) Determine whether the boundary value problem

$$y'' + y' + 2y = 0, \quad y(0) = 0, \quad y(1) = 0$$

is self-adjoint. [7]

SECTION B :

3 a) Deduce the Euler equation relevant to the determination of an extremal for the integral

$$\int_0^1 \left(1+y'^2\right)^{1/2} dx.$$

[6]

b) Determine the stationary functions associated with the integral

$$\int_0^1 (xy'^2 - yy' + y)dx,$$

where a and b are constants, in each of the following situations:

(i) The end conditions $y(0)=0$ and $y(1)=1$ are given. [7]

(ii) Only the end condition $y(0)=0$ is given. [4]

(iii) Only the condition $y(1)=1$ is given. [4]

(iv) No end conditions are given. [4]

4 a) Let $y = 1 + x^2$, where x and y are functions of an independent variable t.

(i) Calculate $\delta \frac{dy}{dx}$ and $\frac{d}{dx}\delta y$ where $x = \frac{1}{t}$ and $\delta x = \varepsilon t^2$. [7]

(ii) Verify the validity of the equation

$$\delta \frac{dy}{dx} = \frac{d}{dx}\delta y - \frac{dy}{dx}\frac{d}{dx}\delta x$$

in this case.

[5]

b) If l is not preassigned, show that the extremals corresponding to the problem

$$\delta \int y'^2 dx = 0; \quad y(0)=2, \quad y(l) = \sin l$$

are of the form $y = 2 + 2x\cos l$, where l satisfies the transcendental equation $2 + 2l\cos l - \sin l = 0$. Also verify that the smallest positive value of l is between $\pi/2$ and $3\pi/4$.

[13]

SECTION C:

5 (i) Consider the boundary value problem

$$y'' = F(x) \quad ; \quad y(0) = 0, \quad y(1) = 0.$$

a) Show that

$$y(x) = \int_0^x (x-t)F(t)dt - x\int_0^1 (1-t)F(t)dt. \qquad [9]$$

b) Show that the integral form of the solution $y(x)$ given in (i) above can be written in the form

$$y(x) = \int_0^1 K(x,t)F(t)dt$$

where $K(x,t)$ is a kernel. [6]

What do we call the integral equation obtained? [1]

(ii) For the initial value problem

$$y'' + xy = 1$$
$$y(0) = y'(0) = 0$$

show that, if $y(x)$ satisfies the initial value problem, then y also satisfies the equation

$$y(x) = \int_0^x t(t-x)y(t)dt + \frac{x^2}{2}. \qquad [9]$$

What do we call this integral equation? [1]

6 a) Transform the problem

$$\frac{d^2y}{dx^2} + y = x, \quad y(0) = 1, \quad y'(1) = 0$$

to a Fredholm integral equation. [8]

b) Verify that the Green's function for the Bessel operator of order n,

$$L[y] = (xy')' - \frac{n^2}{x}y,$$

relevant to the end conditions $y(0) = y(1) = 0$, is of the form

$$G(x,\xi) = \begin{cases} \dfrac{x^n/\xi^n}{2n}(1-\xi^{2n}), & x < \xi \\ \dfrac{\xi^n/x^n}{2n}(1-x^{2n}), & x > \xi, \end{cases}$$

when $n \neq 0$. Hence reduce the problem

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda x^2 - n^2)y = 0, \quad y(0) = y(1) = 0$$

to an integral equation, when $n \neq 0$. [17]