NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 4241

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

SMA 4241: FINANCIAL MATHEMATICS

December 2004

Time : 3 hours

Candidates should attempt **ANY FIVE** questions

A1. (a) Suppose that we have a one-period model with a stock price process X_t; $t = 0, T$ such that $X_0 = 1$ and $X_T = Z$ where Z is a random variable such that $P(Z = 0) = \frac{1}{4}$; $P(Z = 1) = \frac{1}{2}$ and $P(Z = 2) = \frac{1}{4}$.
Investigate the completeness of this market if the risk free interest rate is negligible. [10]

(b) Consider a multi-period, n-dimensional model $\{X_t\}_{t\in\mathcal{T}}$ where $\mathcal{T} = \{0 = t_0, t_1, t_2, \ldots, t_m = T\}$.
Suppose that X_t is a martingale with respect to a measure Q. Let $\theta_t = (\theta_t^{(1)}, \ldots, \theta_t^{(n)})$ be an $\mathcal{F}_t$-adapted, self financing portfolio. Then the fortune gained from this portfolio from time $t = 0$ to time $t = T$ is given by
$V_T - V_0 = \sum_{s=0}^{t_m - 1} \theta_s(X_{s+1} - X_s)$.
Show that $E_Q[V_T - V_0] = 0$. [10]

A2. (a) Show that in a one step binomial tree model of the price of a non dividend paying shares, the risk-neutral probability q of an up movement is given by $q = \frac{1+r-d}{u-d}$ where d, u and r are quantities which you should define. [10]

(b) The market price of a non-dividend paying security with current market price S is being modelled using a one-step binomial tree in which the proportionate changes in the security price following an up and down movement are denoted by u and d respectively. The risk free interest rate over the period is r. Show that if an option on this security has a payoff of z_u following an up movement and a payoff of z_d following a down movement, then the option can be replicated exactly using a portfolio consisting of Δ securities where $\Delta = \frac{z_u - z_d}{(u-d)S}$ and an amount of cash which you should specify. [10]

A3. (a) A European put on a share in an overseas market has one month to expiry and an exercise price of \$2. It is assumed that the share price at expiry will either go down to \$1.50 or up to \$2.50. The risk free interest rate is 1% per month.

(i) Use a hedging argument to value the put option assuming that the underlying share is currently priced at \$2.19. [5]

(ii) Use risk neutral valuation to value the put option assuming that the underlying share is currently priced at \$2.21 [5]

(b) A European call option has a strike price of 105. Find the expected payoff of this option if the share price at expiry can be assumed to be uniformly distributed between 90 and 110. [10]

A4. (a) The table below shows the closing prices (represented by letters) on a particular day for a series of European call options with different strike prices and expiry dates on a particular risky non dividend paying security:

call option prices	Strike 125	Price 150
3 months	W	Y
6 months	X	Z

Write down, with reasons, the strictest inequalities that can be deduced for the relative values of W, X, Y and Z, assuming that the market is arbitrage free. (Your inequalities should not involve other inequalities). [5]

(b) Assume that the current stock price is $S = 80$ and that the proportionate changes in the security price following an upward and a downward movement are u = 1.5 and d = 0.5 respectively. If the risk free interest rate for this period is r = 10% , the stock beta is 0.5 , the market's expected return is 0.2 and the exercise price of a European call option written on the stock is K = 80, calculate the option Greeks, that is, the call option alpha and beta. [15]

A5. Suppose that the continuous time price of a stock is given by

$$S_t = S_0 e^{(\alpha - \frac{1}{2}\sigma^2)t + \sigma B_t}$$

and that with respect to a risk neutral measure Q, which should be given explicitly, we have the stock price as

$$S_t = S_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma B_t}$$

where α, σ and r are constants representing the appreciation rate, the volatility coefficient of the stock and the interest rate respectively and $B_t = B(t, \omega)$ is a standard Brownian motion.

Show that the price of a European call option written on this stock with exercise price K and maturity date T is given by

$C = S_0 N(d_1) - Ke^{-rT} N(d_2)$

where d_1 and d_2 are to be determined explicitly and $N(x)$ is the cumulative distribution of the standard normal distribution. [20]

A6. Suppose that the current share price is $S = 80$ and that for a three period model, the return $\frac{S_t}{S_{t-1}}$ is either u = 1.5 or d = 0.5 for every $t = 1, 2, 3$.

Use the Cox-Ross-Rubinstein model to calculate, at every period, the price of a European call option with exercise price $K = 80$ if the risk free interest rate is r = 10%. [20]

A7. (a) The price of ABC shares is currently 150. In the next period, the price may either go up to 180 or down to 140. An investor buys a covered call on ABC with strike price 150. What is the rational price of this option if interest is 5%? [5]

(b) Assume that the current share price of 260 may either go up to 285 or down to 250 at the end of the year where the annual force of interest is 0.05. Calculate the martingale measure for this market. [5]

Hence or otherwise, calculate the price of a straddle with exercise price 260. [5]

(c) Given that the current share price of ABC shares is S_0 and that in a single period, the price may go up to S^u or down to S^d. If the risk free interest rate is r, where r is constant, calculate, showing all your steps, the price of a European put option with exercise price K written on the stock. Assume that $S^d < K < S^u$. [5]

END OF QUESTION PAPER