NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 4273

DEPARTMENT OF APPLIED MATHEMATICS

QUEUEING THEORY AND STOCHASTIC PROCESSES

MAY 2002

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY THREE** questions from Sections B.

SECTION A: Answer ALL questions in this section [55].

A1. A markov chain has states 1,2,3,4 and 5 and transition probability matrix P. Specify the classes of the chain and determine whether they are, transient or persistant, periodic or aperiodic.

$$P = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

[6]

A2. (a) An organisation has N employees where N is a large number. Each employee has one of three possible job classifications and changes classifications (indipendently) according to a Markov chain with transition probabilities

$$\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$$

What percentage of employees are in each classification at any one point in time. [5]

(b) Three out of every four trucks are followed by a car, while only one out of every five cars is followed by a truck. What fraction of vehicles on the road are trucks? [5]

(c) Consider an irreducible Markov chain with states $0, 1, 2, \cdots, N$. Starting in state i, what is the probability that the process will ever visit state j where $i, j \in \{0, 1, 2, \cdots, N\}$? Explain. [3]

A3. The lifetime of a certain brand of radio is exponentially distributed with a mean of ten years. If Sajanga buys a ten year old radio, what is the probability that it will be working after an additional ten years. [5]

A4. (a) Prove that the property of communication is transitive, that is if $i \leftrightarrow j$ and $j \leftrightarrow k$ then $i \leftrightarrow k$. [6]

(b) prove that in a finite Markov chain not all states are transient. [Hint: Use the limit of $P_{ij}^{(n)}$ as $n \to \infty$ for a transient state and the fact that $\sum_j P_{ij}^{(n)} = 1$]. [10]

A5. A Markov chain has states 1,2,3 with transition probability matrix

$$P = \begin{bmatrix} 1/7 & 2/7 & 4/7 \\ 1/2 & 1/3 & 1/6 \\ 1/2 & 1/4 & 1/4 \end{bmatrix}$$

Obtain the mean recurrence times for all states and the stationary distribution of the system. [15]

SECTION B: Answer THREE questions in this section [45].

B6. Consider a post office with two clerks. Three people, A, B and C, enter the post office at the same time. A and B rush to the clerks, and C waits until either A or B is served. What is the probability that A is still in the post office after the other two have left if

(a) the service time for each clerk is exactly (non-stochastic) ten minutes? [4]

(b) the service times are given by the probability function

$$P(t) = \begin{cases} \frac{1}{3}, & t = 1, 2, 3. \\ 0 & \text{otherwise} \end{cases}$$ [6]

(c) the service times are exponentially distributed with mean $\frac{1}{\mu}$, that is $f(t) = \mu e^{-\mu t}$? [5]

B7. The King of a certain country has a government which contains not more than n members. Each member of the government is dismissed after he has served for a time which is exponentially distributed with parameter μ. From time to time the King selects a citizen from those who are not members of the government. The time between successive selections is exponentially distributed with parameter λ. If the government contains fewer than n members at the time of the citizen's selection then he/she joins the government; otherwise he/she is executed. All selections and dismissals are independent of each other.

(a) Set up forward equations for $p_{ij}(t)$, the probability that the government contains j members at time t given that it contained i members at time 0. [3]

(b) Describe a queueing system which is stochastically equivalent to the above system. [2]

(c) Suppose now that $\lambda = 2\mu$ and $n = 2$. Solve the equations for $P_{02}(t)$. [5]

(c) After the system has been operating for a long time a citizen receives a letter saying that the King has selected him. What is the probability that he will be executed? [5]

B8. Mr. Richest always drives a particular make of car for which the time (t) in years until it first breaks down has probability density function $2e^{-2t}$. In the long run, for what proportion of time is Mr. Richest's car less than six months old if;

(a) he buys a new car as soon as his current car breaks down, [6]

(b) he buys a new car as soon as his current car breaks down, or when his current car is two years old, whichever occurs first. [9]

B9. The probability that an individual in a branching process has k offsprings ($k = 1, 2, 3, \cdots$) is $P_k = q^k p$ where $q = 1 - p \neq 1/2$.

(a) Show that the probability generating function of the size of the n^{th} generation is given by

$$\phi(s) = \frac{p(q^n - p^n) - (q^{n-1} - p^{n-1})pqs}{(q^{n+1} - p^{n+1}) - (q^n - p^n)qs}$$

[7]

(b) Find the mean size of the n^{th} generation. [4]

(c) Find the probability of eventual extinction. [4]

END OF QUESTION PAPER