NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 4273

FACULTY OF SCIENCE

DEPARTMENT OF APPLIED MATHEMATICS

SMA4273: QUEUING THEORY AND STOCHASTIC PROCESSES

SUPPLEMENTARY EXAMINATION

JULY 2003

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY THREE** questions from Sections B.

SECTION A: Answer ALL questions in this section [40].

A1. For the Markov chain whose transition probability matrix is given by P below classify the states as transient or persistent (null or non-null)and as periodic or aperiodic.

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

[10]

A2. Prove that in a finite Markov chain not all states are transient. [5]

A3. Three white and three balck balls are distributed in two urns in such a way that each urn contains three balls regardless of colour. We say that the system is in state $i, i = 0, 1, 2, 3$, if the first urn contains i white balls. At each step, we simultaneously draw one ball from each urn and place it into the other urn (swap them). Let X_n denote the state of the system at time n. Explain why $\{X_n, n = 0, 1, 2, 3 \cdots\}$ is a Markov chain and calculate its transition probability matrix. [10]

A4. Find the limiting(stationary) probability vector for the Markov chain having the transition probability matrix

$$P = \begin{bmatrix} 1/7 & 2/7 & 4/7 \\ 1/2 & 1/3 & 1/6 \\ 1/2 & 1/4 & 1/4 \end{bmatrix}$$

[10]

A5. Let $\{Y_n, n \geq 1\}$ be a sequence of independent random variables with

$$P(Y_n = 1) = p, \text{ and } P(Y_n = -1) = 1 - p$$

Let X_n be defined by

$$X_0 = 0, \ X_{n+1} = X_n + Y_{n+1}$$

Examine whether X_n is a Markov chain. [5]

SECTION B: Answer THREE questions in this section [60].

B6. The matrix below is the transition probability matrix for a Markov chain which involves tansitions between states 0,1,2,3.

$$P = \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix}$$

(a) Find the first return probabilities for state 0, $f_{00}^{(n)}$ for n=1,2,3,4,5. [10]

(b) Find the mean recurrence times for the states 0 and 2. [10]

B7. A garage's demand for carburetors each day is a random variable which takes values 0,1 or 2 with probabilities 0.3, 0.5, 0.2 respectively, each day's demand being independent of that of other days. If at the end of a day the garage's stock of carburetors is less than 2, it is made up overnight to a level of 3.(*If the stock at the end of a day is not less than 2 then there is no re-stocking*). Let X_n be the level of stock at the end of day n.

(a) Determine the transition probability matrix for $\{X_n\}$. [8]

(b) If the stock at the end of a particular day is zero, what is the probability that the stock is again zero three days later? [12]

B8. (a) A fly walks along an equilateral triangle ABC. When it reaches a vertex it chooses which edge to walk along next with equal probabilities. If it starts at A, find the mean number of edges it walks along until it comes back to A(mean recurrent time for state A). [9]

(b) The matrices given below are transition probability matrices for two Markov chains with states {0,1,2};

$$\text{(i)} \begin{bmatrix} p_1 & q_1 & r_1 \\ 0 & p_2 & q_2 \\ q_3 & 0 & p_3 \end{bmatrix} \qquad \text{(ii)} \begin{bmatrix} 0 & 1/3 & 2/3 \\ 0 & 1 & 0 \\ 1/4 & 3/4 & 0 \end{bmatrix}$$

For matrix (i) find the first return/passage probabilities $f_{00}^{(n)}$ and $f_{01}^{(n)}$ for $n = 1, 2, 3$ [6]

For matrix (ii) classify the states and state the class properties they have. [5]

B9. (a) A counting process $N(t)$ is Poisson process with rate λ if it has independent and stationary increments, and if in addition $N(0) = 0$;

$Pr[N(\delta t) = 1] = \lambda\delta t + o(\delta t)$; $Pr[N(\delta t) = 2] = o(\delta t)$ as $\delta t \to 0$

Using these conditions show that

$$Pr[N(t) = 1] = \lambda t e^{-\lambda t}$$

[10]

(b) Patients arrive at a hospital emergency room at times which form a Poisson process with rate λ per hour. Each patient's condition is independently either life threatening or not with probabilities α and $1 - \alpha$ respectively. The emergency room's consultant has to attend a meeting which lasts for a time which is uniformly distributed on the interval [1,2] hours.

Show that the probability that exactly one patient with a life threatening condition arrives during a period of length t hours is

$$\alpha\lambda t e^{-\alpha\lambda t}.$$

[10]

END OF QUESTION PAPER