NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 4273

FACULTY OF SCIENCE

**DEPARTMENT OF APPLIED MATHEMATICS**

QUEUING THEORY AND STOCHASTIC PROCESSES

MAY 2005

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY THREE** questions from Sections B.

**SECTION A: Answer ALL questions in this section [40].**

**A1.** The probability generating function of a random variable $X$ with $P(X = x) = p_x$ is given by $g(s) = \sum p_x s^x$.
If $x$ is a random variable with probability generating function $g(s)$ prove that:

(a) $E(x) = g'(1)$. [3]

(b) $VarX = g'(1) + g''(1) - [g'(1)]^2$. [5]

**A2.** The matrix below is the transition probability matrix for a Markov Chain which involves transitions between states 0,1,2,3. Find the first return probabilities for sattes 0, $f_{00}^n$ for n=1,2,3,4,5,6.

$$P = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

[5]

**A3.** Given that $i$ and $j$ are persistent and that $i \leftrightarrow j$ show that if $i$ is periodic then $j$ is also periodic with the same period. [5]

**A4.** What do you understand by the following terms:

(a) Stochastic process. [2]

(b) Ergordic Markov Chain. [2]

**A5.** Given a branching process with the following offspring distributions find the mean number of offsprings produced by an individual, $\mu$, and determine the extinction probability $s$.

(a) $p_0 = 0.25$, $p_1 = 0.4$ and $p_2 = 0.35$. [3]

(b) $p_0 = 0.5$, $p_1 = 0.25$ and $p_2 = 0.25$. [3]

**A6.** Explain the following queueing systems:

(a) $(M/M/1) : (FIFO/\infty/\infty)$. [3]

(b) $(M/E_k/1) : (SIRO/N/M)$. [3]

(c) $(M/M/c) : (FIFO/\infty/\infty)$. [3]

(d) $(GI/G/c) : (GD/N/M)$. [3]

**SECTION B: Answer THREE questions in this section [60].**

**B7.** A Markov chain has state space $S = \{1, 2, 3, 4, 5, 6, \}$ with probability transition matrix:

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

(a) Identify the classes in the chain and classify them as transient or persistant, periodic or aperiodic. [5]

(b) Show that the first recurrence probability in $n$ steps for state 1 is given by

$$f_{11}^{(n)} = \begin{cases} 1/2 & \text{for n=1} \\ \frac{1}{8}\left(\frac{3}{4}\right)^{n-2} & \text{for } n \geq 2 \end{cases}$$

[5]

(c) Show that the mean recurrence time for state 1 is $\mu_{11} = 3$. [5]

(d) Find the limiting(stationary) probability vector for the Markov Chain having the transition probability matrix:

$$P = \begin{bmatrix} 1/7 & 2/7 & 4/7 \\ 1/2 & 1/3 & 1/6 \\ 1/2 & 1/4 & 1/4 \end{bmatrix}$$

[5]

**B8.** Two gamblers A and B play a succession of independent games against each other. For each game, either A or B wins and the loser pays the winner \$1. Player A has probability $p = \frac{1}{(1+\beta)}$ of winning each game ($\beta \neq 1, \beta > 0$) and the probability $1-p$ of loosing the game. Player A starts with \$a and player B with \$b. The game stops when either A or B is ruined.

(a) Show that

$$\pi_{r+1} - (1-\beta)\pi_r + \beta\pi_{r-1} = 0, \quad for \quad r = 2, 3, ..., a+b-2;$$

Where $\pi_r$, is the probability that A is eventually ruined given that he currently has \$r. Hence or otherwise find the probability that it is A who is ruined. [5]

(b) Show that the mean number of games until the series ends is:

$$\left(\frac{1+\beta}{1-\beta}\right)\left[\left(\frac{a+b}{1-\beta^{a+b}}\right)(1-\beta^a) - a\right].$$

[10]

(c) A random walk has the following probabilities

$$p_{i,i+2} = p, \qquad p_{i,i-1} = 1-p.$$

Write the transition probability matrix for the random walk and determine if any of the states in the random walk are transient. [5]

**B9.** People arrive at a certain gathering at intervals that are exponentially distributed with parameter $\lambda$, that is the inter-arrival times have density function given by $f(x) = \lambda e^{-\lambda x}$. Let $N(t)$ be the number of arrivals by time $t$. Show that $N(t)$ has the Poisson distribution with mean $\lambda t$, that is $Pr[N(t) = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$. [20]

**B10.** (a) A machine shop has six machines and two mechanics to repair faulty machines. Each machine breaks down in a poisson fashion with a mean of 2 per hour. The repair time on each machine follows an exponential distribution with mean of 20 minutes.

(i) What is the probability that the two mechanics are idle? [3]

(ii) What is the expected number of idle mechanics? [4]

(iii) What is the expected number of faulty machines waiting for repair? [3]

(b) Two telephone cables can each handle one message at a time. The breakdown for each has an exponential distribution with parameter $\lambda$. The time to repair each cable has the exponential distribution with parameter $\mu$. Two repairmen are available, one for each machine. If state $i$ corresponds to $i$ machines undergoing repairs ($i = 0, 1, 2$), set up but do not solve, the Kolmolgorov backward equations connecting $P_2(t)$, $P_1(t)$ and $P_0(t)$. [10]

**END OF QUESTION PAPER**