NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 4273

FACULTY OF SCIENCE

DEPARTMENT OF APPLIED MATHEMATICS

QUEUING THEORY AND STOCHASTIC PROCESSES

JULY 2005

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY THREE** questions from Sections B.

SECTION A: Answer ALL questions in this section [40].

A1. A random walk has the following probabilities

$$p_{i,i+2} = p, \qquad p_{i,i-1} = 1 - p.$$

Write the transition probability matrix for the random walk and determine if any of the states in the random walk are transient. [5]

A2. The matrix below is the transition probability matrix for a Markov Chain which involves transitions between states 0,1,2,3. Find the first return probabilities for states 0, f_{00}^n for n=1,2,3,4,5,6.

$$P = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

[5]

A3. Prove that in a finite Markov chain not all states are transient. [5]

A4. What do you understand by the following terms:

(a) Stochastic process. [2]

(b) Ergordic Markov Chain. [2]

A5. Given a branching process with the following offspring distributions find the mean number of offsprings produced by an individual,μ, and determine the extinction probability s.

(a) $p_0 = 0.25, \quad p_1 = 0.4$ and $p_2 = 0.35$. [3]

(b) $p_0 = 0.5, \quad p_1 = 0.25$ and $p_2 = 0.25$. [3]

A6. (a) What does $(M/M/c) : (GD/N/\infty)$ mean? [3]

(b) A fast food restaurant has one drive-in window. Cars arrive according to a poisson distribution at the rate of 2 cars every 5 minutes. The space infront of the window can accomodate at most 10 cars, including the one being served. Other cars can wait outside this space if necessary. The service time per customer is exponential with mean 1.5 minutes. Determine the following:

(i) The probality that the facility is idle. [3]

(ii) The expected number of customers waiting to be served. [3]

(iii) The expected waiting time until a customer reaches the window to place an order. [3]

SECTION B: Answer THREE questions in this section [60].

B7. (a) A service station has one petrol pump. Cars wanting petrol arrive according to a Poisson process at a mean rate of 15 per hour. However, if the pump is already being used these cars balk (drive to another service station). In particular, if there are n cars already at the service station, the probability that an arriving potential customer will balk is $\frac{n}{3}$ for n =1,2,3. The time required to service a car has an exponential distribution with a mean of 4 minutes.

(i) Construct the rate diagram for this queueing system. [5]

(ii) Develop the balance equations. [5]

(b) Derive the equation

$$P_n = \left(\frac{\lambda_{n-1}}{\mu_n}\right) P_{n-1}$$

for the steady state probability P_n of n customers in a queueing system, stating the conditions under which it is valid, and explain its importance. [5]

(c) A petrol station has two petrol pumps and space for three cars to queue. Customers arrive at random at a mean rate of one every minute. Customers who arrive to find no queueing space leave and do not return. Service times are exponentially distributed with a mean of 3 minutes. It is proposed that a third petrol pump be installed, leaving space for 2 cars to queue. The petrol station is open for 40 hours per week, should the third pump be installed?(You may ignore the cost of installing the third pump). [5]

B8. People arrive at a certain gathering at intervals that are exponentially distributed with parameter λ, that is the inter-arrival times have density function given by $f(x) = \lambda e^{-\lambda x}$. Let $N(t)$ be the number of arrivals by time t. Show that $N(t)$ has the Poisson distribution with mean λt, that is $Pr[N(t) = n] = \dfrac{e^{-\lambda t}(\lambda t)^n}{n!}$. [20]

B9. (a) A fly walks along an equilateral triangle ABC. When it reaches a vertex it chooses which edge to walk along next with equal probabilities. If it starts at A, find the mean number of edges it walks along until it comes back to A(mean recurrent time for state A). [9]

(b) The matrices given below are transition probability matrices for two Markov chains with states {0,1,2};

$$\text{(i)} \begin{bmatrix} p_1 & q_1 & r_1 \\ 0 & p_2 & q_2 \\ q_3 & 0 & p_3 \end{bmatrix} \qquad \text{(ii)} \begin{bmatrix} 0 & 1/3 & 2/3 \\ 0 & 1 & 0 \\ 1/4 & 3/4 & 0 \end{bmatrix}$$

For matrix (i) find the first return/passage probabilities $f_{00}^{(n)}$ and $f_{01}^{(n)}$ for $n = 1, 2, 3$ [6]

For matrix (ii) classify the states and state the class properties they have. [5]

B10. A Markov chain has state space $S = \{1, 2, 3, 4, 5, 6, \}$ with probability transition matrix:

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

(a) Identify the classes in the chain and classify them as transient or persistant, periodic or aperiodic. [5]

(b) Show that the first recurrence probability in n steps for state 1 is given by

$$f_{11}^{(n)} = \begin{cases} 1/2 & \text{for n=1} \\ \frac{1}{8}\left(\frac{3}{4}\right)^{n-2} & \text{for } n \geq 2 \end{cases}$$

[5]

(c) Show that the mean recurrence time for state 1 is $\mu_{11} = 3$. [5]

(d) Find the limiting(stationary) probability vector for the Markov Chain having the transition probability matrix:

$$P = \begin{bmatrix} 1/7 & 2/7 & 4/7 \\ 1/2 & 1/3 & 1/6 \\ 1/2 & 1/4 & 1/4 \end{bmatrix}$$

[5]

END OF QUESTION PAPER