NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 5181

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

SMA 5181: STOCHASTIC DIFFERENTIAL EQUATIONS

DECEMBER 2001

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY THREE** questions from Section B.

SECTION A: [25 marks]

Answer **ALL** questions from this section.

A1. Let $\mathcal{F}_t$ denote some filtration such that the Brownian motion $B(t,\omega)$ is $\mathcal{F}_t$ adapted. Calculate the following conditional expectations:

(a) $E\left[\left(B(s)+B^2(t)\right)e^{B(t)}|\mathcal{F}_t\right]$, $s \leq t$ [3]

(b) $E\left[B^2(s)e^{B(t)-B(s)}|\mathcal{F}_t\right]$, $s \leq t$ [3]

A2. (a) Let $Y(\omega)$ satisfy $E[|Y|] < \infty$ and define M_t by $M_t = E[Y|\mathcal{F}_t]$. Prove that M_t is an $\mathcal{F}_t$-martingale. [3]

(b) If $X_t : \Omega \to \mathbb{R}^n$ is a stochastic process, let $\mathcal{H}_t = \mathcal{H}_t^X$ denote the σ−algebra generated by $\{X_s(.);\ s \leq t\}$

(i) Show that if X_t is a martingale with respect to some filtration $\{N_t\}_{t\geq 0}$, then X_t is also a martingale with respect to its own filtration $\{\mathcal{H}_t^X\}_{t\geq 0}$. [3]

(ii) Show that if X_t is a martingale with respect to $\mathcal{H}_t^X$, then $E[X_t] = E[X_0]$ for all $t \geq 0$ [3]

A3. (a) Prove that every stopping time is optional and that the two concepts coincide if the filtration is right-continuous. [4]

(b) Suppose that we have the two continuous semi-martingales

$$X_t = X_0 + M_t + A_t$$
$$Y_t = Y_0 + N_t + C_t \quad 0 \leq t < \infty$$

where M and N are continuous (local) martingales and A_t, C_t are adapted, continuous processes of bounded variation with $B_0 := C_0 = 0$ almost surely. Prove the integration by parts formula

$$\int_0^t X_s dY_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s dX_s - < M, N >_t$$

where $< P, Q >$ denotes the cross variation of P and Q [3]

(c) Use Ito's formula to prove that $\int_0^t B_s^2 dB_s = \frac{1}{3} B_t^3 - \int_0^t B_s ds$ [3]

SECTION B: [75 marks]

Answer **ANY THREE** questions from this section. Each question carries 25 marks.

B4. (a) Let X_t be an Ito-integral $dX_t = v(t, \omega) dB_t(\omega)$ where $v \in \mathbb{R}^n$, $B_t \in \mathbb{R}^n$

(i) Show that X_t^2 is not in general a martingale [5]

(ii) Prove that $M_t := X_t^2 - \int_0^t |v(s)|^2 ds$ is a martingale. [5]

(b) Let $(B_1, \ldots, B_n)$ be a Brownian motion in R^n, $\alpha_1, \ldots, \alpha_n$ constants. Solve the stochastic differential equation

$$dX_t = rX_t dt + X_t \left(\sum_{k=1}^{n} \alpha_k dB_k(t) \right), \; r \text{ constant and } X_0 > 0$$ [5]

(c) Solve the Ornstein-Uhlenbeck equation $dX_t = \mu X_t dt + \sigma dB_t$ where μ and σ are real constants and $B_t \in \mathbb{R}$ is Brownian motion. [4]

Hence find $E[X_t]$ and $Var[X_t] = E[X_t - E^2[X_t]]^2$ where X_t is the Ornstein-Uhlenbeck process. [6]

B5. (a) Consider an n-dimensional Brownian motion $B := (B_1, \ldots, B_n)$ starting at $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ $(n \geq 1)$ and assume $|a| < R$. What is the expected value of the first exist time τ_k of B from the ball $K = K_R = \{x \in \mathbb{R}^n : |x| < R\}$? [6]

(b) Let $X_t = x + B_t$ be the Brownian motion starting at x. Assume that $f(y) = y^2$. Show that X_t satisfies the Markov property $E^x[f(X_{t+h})|\mathcal{F}_h] = E^{X_h(\omega)}[f(X_t)]$ where h is a positive real constant. [9]

(c) Find the generator of the Ito-diffusion

$$dX_t = \begin{bmatrix} dX_1(t) \\ dX_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ 0 & X_1 \end{bmatrix} \begin{bmatrix} dB_1 \\ dB_2 \end{bmatrix}$$ [5]

(d) Find a diffusion X_t with generator $Af(x) = f'(x) + f''(x)$ [5]

B6. (a) State the variational inequalities for optimal stopping [2]

(b) Let the price of a stock be given by the geometric Brownian motion $dS(t) = S(t)\,[rdt + \sigma dB(t)]$, where r and σ are constants. When is the right time to sell the stock? [12]

(c) Solve the optimal stopping problem $\gamma(x) = \sup_{\tau} E^{(s,x)}\left[e^{-\rho T} B_\tau^2\right]$. [11]

B7. (a) State and prove the Hamilton-Jacobi-Bellman (HJB) theorem. [11]

(b) Let X_t denote the wealth of a person at time t. Suppose that the person has the choice of two different investments. The price $p_1(t)$ at time t of one of the assets (the risky asset) is assumed to satisfy the equation

$\frac{dp_1}{dt} = p_1(a + \alpha W_t)$ where W_t denotes white noise and $a, \alpha > 0$ are constants measuring the average relative rate of change of p_1 and the size of the noise respectively. The price p_2 of the other asset (the risk free asset) satisfies a similar equation, but with no noise

$dp_2 = bp_2 dt$.

At each instant, the person can choose how big fraction u of his wealth X_t, $t \geq 0$ he will invest in the risky assest, thereby investing the fraction $1 - u$ in the safe one. Suppose that the person's starting capital is $X_t = x$ at time $t > 0$ and that his utility function is $N(x) = x^r$, $0 < r < 1$.

Solve the stochastic control problem

$\Phi(s, x) = \sup\{J^u(s, x);\ u \text{ is Markov control},\ 0 \leq u \leq 1\} = J^{u^*}(s, x)$ where $J^u(s, x) = E^{s,x}\left[N(X_T^u)\right]$. [14]

B8. (a) Solve the stochastic differential equation $dX_t = (m - X_t)dt + \sigma e^{-2t} dB_t$; $X_0 = x$. [10]

(b) Let X_t denote the solution of (a). Compute $E[X_t]$ and $Var[X_t]$. [10]

(c) Prove that $\lim_{t\to\infty} X_t$ exists as in L^2- limit and compute this value. [5]

END OF QUESTION PAPER