NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF APPLIED MATHEMATICS

APRIL 2003 EXAMINATIONS

SMA5192 MATHEMATICAL MODELLING II

Full marks will be gained by answering THREE questions

1. Consider a spring pendulum.

   Derive the equations of the free motion of the spring pendulum using

   (a) Lagrangian formulation,

   (b) Vectorial method, and

   (c) Hamiltonian method.

2. (a) What do you regard as the essential difference between natural convection problem and forced convection problem? Explain the physical significance of the non-dimensional groupings termed the Rayleigh number Ra, Reynolds number Re, Grashof number Gr and Prandtl number Pr. Illustrate for $Pr << 1$, and $Pr >> 1$, by sketching thermal and velocity profiles, the role that the Prandtl number plays in determining the size of the thermal boundary layer thickness δ_T, the viscous boundary layer thickness δ_{visc} and the velocity penetration depth δ_{vel}. Choose a simple configuration characterising each type of convection problem.

   (b) A model of a single phase heat exchanger in an industrial heat exchanger involves a steady and fully-developed one-dimensional shear flow

   $$\vec{u} = [u, v, w] = [U_0 z/d, 0, 0]$$

   between the planar surfaces $z = 0$ and $z = d$ generated by the uniform motion at a speed U_0 of the upper plane which is maintained at an ambient temperature T_0. The incompressible fluid is assumed isothermal and at temperature T_0 for $x < 0$ but the lower plane $z = 0$ has temperature $T_1 > T_0$ for $x > 0$. The development in space of the thermal profile is then governed by the energy equation

   $$u\frac{\partial T}{\partial x} = k[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}], \quad x > 0, \quad 0 < z < d,$$

   where k is the thermal diffusivity.

   (i) By making the transformations

   $$x = dX, \quad z = dZ, \quad T = T_0 + (T_1 - T_0)\theta$$

   determine the equation and boundary conditions for the non-dimensional temperature θ. Hence define the Peclet number and explain its physical significance.

(ii) Explain why the thermal profile is expected to tend to $\theta_s(X,Z) = 1 - Z$ as $X \to \infty$. Hence determine the equation and boundary conditions satisfied by the deviation temperature $\Theta(X,Z) = \theta + Z - 1$. On assuming an exponential decay in X of Θ show that the equation is separable and therefore that θ has the form

$$\theta(X,Z) = 1 - Z + \sum_{n=1}^{\infty} a_n e^{-\lambda_n X} B_n(Z).$$

State clearly the differential eigenvalue problem satisfied by the functions $B_n(Z)$ with associated eigenvalues λ_n. How can the constants a_n be specified, and is it possible to make θ and $\partial\theta/\partial X$ continuous at $X = 0$?

3. The equation for the vibration of a two-dimensional plate was obtained in 1811 by Lagrange, and can be shown to be:

$$D[\frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4}] + \rho h \frac{\partial^2 w}{\partial t^2} = 0 \qquad (1)$$

where $D = \frac{Eh^2}{12(1-\sigma)}$ is a constant, ρh is mass of unit square.

(a) Discuss the boundary conditions of the vibration of the rectangular plate when: (i) the edges are fixed, (ii) the edges are hinged and (iii) when the edges are free.

(b) By looking for a solution of the form

$$W(x,y,t) = w(x,y)\sin(\omega t + \alpha)$$

discuss the solutions to the eigenvalue problem problem

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} - \lambda^4 w = 0 \qquad (2)$$

For three possible sets of boundary conditions.

4. The fluid dynamical equations of a combustible system with reactive processes are:

$$D\rho/Dt + \mu\rho U_x = 0$$
$$\rho DU/Dt + \mu/\rho P_x = 0$$
$$DT/Dt + \mu(\gamma - 1)TU_x = -QDY/DT$$
$$\rho DY/Dt = [-\epsilon\rho Y/Q]exp[1/\epsilon(1 - 1/T)]$$
$$P = RT$$

where D/Dt is defined to be $\partial/\partial t + \mu U \partial/\partial x$. The five dependent variables ρ, U, P, T, Y, represent density, velocity, pressure, temperature, and mass fraction (concentration), while ϵ, Q and γ represent the activation energy, the chemical heat release measure and the ratio of specific heats.

By seeking asymptotic solutions of the form

$$\rho = 1 + \epsilon(\rho_0 + \epsilon^{1/2}\rho_1 + \rho_2)$$

$$P = 1 + \epsilon(P_0 + \epsilon^{1/2}P_1 + P_2)$$
$$T = 1 + \epsilon(\phi_0 + \epsilon^{1/2}\phi_1 + \phi_2)$$
$$Y = 1 + \epsilon(Y_0 + \epsilon^{1/2}Y_1 + Y_2)$$
$$U = \epsilon(U_0 + \epsilon^{1/2}U_1 + U_2)$$

find the leading order terms of the expansions ρ_0, ϕ_0, P_0, Y_0 and U_0 subject to the initial conditions:

$$\rho_0 = \rho_i(x), \quad U_0 = U_i(x), \quad P_0 = P_i(x), \quad \phi_0 = \phi_i(x), \quad Y_0 = Y_i(x).$$

5. The nature of both **field** and **constitutive** equations determines the degree of complexity in the theory of electrohydrodynamic interactions in liquid dielectrics. Explain the difference between these two distinct sets of equations.

   When an electrical dsicharge occurs from a current carrying high voltage electrode immersed in "insulating" transformer oil a front of charge is propagated through the static liquid torwards the earthed collector. However after time τ_s the fluid losses its stability and becomes turbulent behind this charge front and the current surges to a substantial value before circuit breakers shut down the supply of current. This situation is to be modelled by a transient injection of unipolar charge of invariant mobility κ into a static liquid dielectric layer of depth d and constant permittivity ϵ. The governing one-dimensional equations are taken to be

$$\partial Q/\partial t + \partial j/\partial x = 0, \quad \epsilon \partial E/\partial x = Q, \quad j = \kappa Q E, \quad E = -\partial V/\partial x$$

   where$Q(x,t)$, $j(x,t)$, $E(x,t)$, and $V(x,t)$ are the charge density, ionic current, electric field and potential respectively. The associated boundary conditions are:

$$V(0,t) = V_0, \quad V(d,t) = 0, \quad Q(0,t) = Q_0(t) \ \ for \ \ t > 0$$

   and the initial conditions are

$$E(x,0) = V_0/d, \quad Q(x,0) = 0, \ \ for \ 0 < x < d,$$

   where V_0 is a constant.

   (a) Explain the significance of each of these equations and the boundary conditions.

   (b) Show that the total current $J = j + \epsilon \partial E/\partial t$ is independent of x and is, in a frame moving with the front velocity κE, the time rate of change of the isotropic displacement field ϵE. By integrating $J(t)$ over the layer obtain the equations

$$\frac{\partial E(0,t)}{\partial t} = \frac{\kappa}{2d}[E^2(d,t) - E^2(0,t)] - \frac{\kappa}{\epsilon}Q_0(t)E(0,t)$$

$$\frac{\partial E(d,t)}{\partial t} = \frac{\kappa}{2d}[E^2(d,t) - E^2(0,t)]$$

for time $t < \tau$ where τ is the transient time of the ion front defined by

$$d = \int_0^{\tau} \kappa E(d,t)dt$$

(c) Assuming the space-charge-limited situation, for which $Q_0(t) \to \infty$, $E(0,t) \to 0$ for all t such that $Q_0(t)E(0,t)$ remains finite, determine $E(d,t)$ and hence show that

$$\tau = \frac{2d^2}{\kappa V_0}(1 - e^{-1/2}).$$

(d) At time $t < \tau$, the charge front has coordinate $\lambda(t)$ and the electrical potential $V[\lambda(t)]$ is given by

$$V[\lambda(t)] = [d - \lambda(t)]E(d,t) = [d - \int_0^t \kappa E(d,s)ds]E(d,t).$$

If, however, the fluid becomes unstable as soon as $V_0 - V(\lambda) = V_{crit} << V_0$ in a time $\tau_s << \tau$ show that

$$\tau_s \approx \frac{2d^2 V_{crit}}{\kappa V_0^2}.$$