NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF APPLIED MATHEMATICS

JULY 2003 SUPPLEMENTARY EXAMINATIONS

SMA5192 MATHEMATICAL MODELLING II

Full marks will be gained by answering THREE questions

1. (a) Obtain a one-term composite expansion for the solution of

$$\epsilon d^2 f/dx^2 + df/dx + f/(1+x) = 2, \quad 0 < x < 1, \quad 0 < \epsilon \leq 1$$

with boundary conditions $f(0) = 0$, $f(1) = 3$, using Prandtl's matching criteria given that a boundary layer exists at $x = 0$.

(b) Consider the general linear equation

$$\epsilon d^2 f/dx^2 + a(x) df/dx + b(x) f = c(x), \quad x_1 < x < x_2, \quad 0 < \epsilon \leq 1$$

with boundary conditions $f(x_1) = A$, $f(x_2) = B$. Show that if $a(x) > 0$ for $x_1 < x < x_2$ a boundary layer will occur at $x = x_1$ and if $a(x) < 0$ a boundary layer will occur at $x = x_2$.

2. Consider a thermally induced convection of a fluid system. The equations governing the velocity field v, temperature field T and pressure p can be written as:

$$\rho(\partial v/\partial t + v \cdot \nabla v) = -\nabla p + \rho\nu\nabla^2 v + \rho[1 - \alpha(T - T_0)]g$$
$$\nabla \cdot v = 0$$
$$\partial T/\partial t + v \cdot \nabla T = \kappa \nabla^2 T$$

where ρ is the fluid density at reference temperature T_0, ν the kinematic viscousity and κ the thermal diffusivity.

(a) Show that the above equations admit a one-dimensional hydrostatic pure thermal conduction solution given by

$$v = v_e = 0, \quad T = T_e(z) = T_0 - \frac{\Delta T \cdot z}{d}, \quad p = p_e(z)$$

when a temperature difference $\Delta T^o C$ is applied to a fluid between planes $z = 0$ (below) and $x = d$ (above). Hence obtain an expression for the equilibrium pressure distribution $p_e(z)$.

(b) If this hydrostatic equilibrium is disturbed then a non-zero fluid velocity and a no longer linear temperature profile will result. These perturbations will in general depend upon x, y, z and t. By writing $v = v_e + u$, $T = T_e(z) + \theta$ and $p = p_e + P$, show that the perturbations $u(x, y, z, t)$, $\theta(x, y, z, t)$ and $P(x, y, z, t)$ are governed by equations

$$\rho(\frac{\partial u}{\partial t} + u \cdot \nabla u) = -\nabla P - \alpha\rho\theta g + \rho\nu\nabla^2 u$$

$$\nabla u = 0$$

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla\theta = \kappa\nabla^2\theta - u\nabla T_e.$$

(c) However, for simplicity we can confine our attention to two-dimensional disturbances and consequently reduce the investigations to two coupled nonlinear partial differential equations for a stream function $\phi(x, z, t)$ and temperature $\theta(x, z, t)$.

(i) Show that it is possible to write in terms of $\phi(x, z, t)$

$$u = \nabla \times (\phi\hat{j}) = [-\frac{\partial \phi}{\partial z}, 0, \frac{\partial \phi}{\partial x}],$$

so that the continuity equation is automatically satisfied.

(ii) Show that the energy equation derived in above in $2(b)$ can be written as

$$\frac{\partial \theta}{\partial t} = \frac{\partial(\phi, \theta)}{\partial(x, z)} + \frac{\Delta T}{d} \cdot \frac{\partial \phi}{\partial x} + \kappa\nabla^2\theta.$$

(iii) On using the vector identity $\nabla^2 \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$ and operating the Navier-Stokes equation derived in $2(b)$ the operator $\hat{j} \cdot \nabla\times$ (that is, the $y - th$ component of the curl of the equation) deduce that

$$\frac{\partial \nabla^2\phi}{\partial t} = -\frac{\partial(\phi, \nabla^2\phi)}{\partial(x, z)} + \nu\nabla^4\phi + \alpha g\frac{\partial \theta}{\partial x}.$$

3. The equation for the vibration of a two-dimensional plate was obtained in 1811 by Lagrange, and can be shown to be:

$$D[\frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4}] + \rho h\frac{\partial^2 w}{\partial t^2} = 0 \qquad (1)$$

where $D = \frac{Eh^2}{12(1-\sigma)}$ is a constant, ρh is mass of unit square.

(a) Discuss the boundary conditions of the vibration of the rectangular plate when: (i) the edges are fixed, (ii) the edges are hinged and (iii) when the edges are free.

(b) By looking for a solution of the form

$$W(x, y, t) = w(x, y)\sin(\omega t + \alpha)$$

discuss the solutions to the eigenvalue problem problem

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} - \lambda^4 w = 0 \qquad (2)$$

For three possible sets of boundary conditions.

4. The fluid dynamical equations of a combustible system with reactive processes are:

$$D\rho/Dt + \mu\rho U_x = 0$$
$$\rho DU/Dt + \mu/\rho P_x = 0$$
$$DT/Dt + \mu(\gamma - 1)TU_x = -QDY/DT$$
$$\rho DY/Dt = [-\epsilon\rho Y/Q]exp[1/\epsilon(1 - 1/T)]$$
$$P = RT$$

where D/Dt is defined to be $\partial/\partial t + \mu U \partial/\partial x$. The five dependent variables ρ, U, P, T, Y, represent density, velocity, pressure, temperature, and mass fraction (concentration), while ϵ, Q and γ represent the activation energy, the chemical heat release measure and the ratio of specific heats.

By seeking asymptotic solutions of the form

$$\rho = 1 + \epsilon(\rho_0 + \epsilon^{1/2}\rho_1 + \rho_2)$$
$$P = 1 + \epsilon(P_0 + \epsilon^{1/2}P_1 + P_2)$$
$$T = 1 + \epsilon(\phi_0 + \epsilon^{1/2}\phi_1 + \phi_2)$$
$$Y = 1 + \epsilon(Y_0 + \epsilon^{1/2}Y_1 + Y_2)$$
$$U = \epsilon(U_0 + \epsilon^{1/2}U_1 + U_2)$$

find the leading order terms of the expansions ρ_0, ϕ_0, P_0, Y_0 and U_0 subject to the initial conditions:

$$\rho_0 = \rho_i(x), \quad U_0 = U_i(x), \quad P_0 = P_i(x), \quad \phi_0 = \phi_i(x), \quad Y_0 = Y_i(x).$$

5. The nature of both **field** and **constitutive** equations determines the degree of complexity in the theory of electrohydrodynamic interactions in liquid dielectrics. Explain the difference between these two distinct sets of equations.

When an electrical dsicharge occurs from a current carrying high voltage electrode immersed in "insulating" transformer oil a front of charge is propagated through the static liquid torwards the earthed collector. However after time τ_s the fluid losses its stability and becomes turbulent behind this charge front and the current surges to a substantial value before circuit breakers shut down the supply of current. This situation is to be modelled by a transient injection of unipolar charge of invariant mobility κ into a static liquid dielectric layer of depth d and constant permittivity ϵ. The governing one-dimensional equations are taken to be

$$\partial Q/\partial t + \partial j/\partial x = 0, \quad \epsilon\partial E/\partial x = Q, \quad j = \kappa QE, \quad E = -\partial V/\partial x$$

where$Q(x,t)$, $j(x,t)$, $E(x,t)$, and $V(x,t)$ are the charge density, ionic current, electric field and potential respectively. The associated boundary conditions are:

$$V(0,t) = V_0, \quad V(d,t) = 0, \quad Q(0,t) = Q_0(t) \quad for \quad t > 0$$

and the initial conditions are

$$E(x,0) = V_0/d, \quad Q(x,0) = 0, \; for \; 0 < x < d,$$

where V_0 is a constant.

(a) Explain the significance of each of these equations and the boundary conditions.

(b) Show that the total current $J = j + \epsilon \partial E/\partial t$ is independent of x and is, in a frame moving with the front velocity κE, the time rate of change of the isotropic displacement field ϵE. By integrating $J(t)$ over the layer obtain the equations

$$\frac{\partial E(0,t)}{\partial t} = \frac{\kappa}{2d}[E^2(d,t) - E^2(0,t)] - \frac{\kappa}{\epsilon}Q_0(t)E(0,t)$$

$$\frac{\partial E(d,t)}{\partial t} = \frac{\kappa}{2d}[E^2(d,t) - E^2(0,t)]$$

for time $t < \tau$ where τ is the transient time of the ion front defined by

$$d = \int_0^\tau \kappa E(d,t)dt$$

(c) Assuming the space-charge-limited situation, for which $Q_0(t) \to \infty$, $E(0,t) \to 0$ for all t such that $Q_0(t)E(0,t)$ remains finite, determine $E(d,t)$ and hence show that

$$\tau = \frac{2d^2}{\kappa V_0}(1 - e^{-1/2}).$$

(d) At time $t < \tau$, the charge front has coordinate $\lambda(t)$ and the electrical potential $V[\lambda(t)]$ is given by

$$V[\lambda(t)] = [d - \lambda(t)]E(d,t) = [d - \int_0^t \kappa E(d,s)ds]E(d,t).$$

If, however, the fluid becomes unstable as soon as $V_0 - V(\lambda) = V_{crit} << V_0$ in a time $\tau_s << \tau$ show that

$$\tau_s \approx \frac{2d^2 V_{crit}}{\kappa V_0^2}.$$