National University of Science and Technology
Faculty of Applied Sciences

Department of Applied Mathematics

December 2002 Examination

Numerical Solution of PDEs SMA 5261

Maximum time allowed : **3 Hours**
Answer any **four** questions. Each question carries 25 marks.

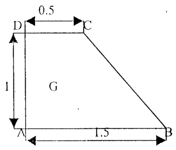
1. Approximate the solution to the parabolic initial boundary value problem

$$u_t = u_{xx} + 1 \text{ for } 0 < x < 1, t > 0;$$
$$u(x,0) = 0 \text{ for } 0 < x < 1,$$
$$u_x(0,t) - 0.5u(0,t) = 0 \text{ and } u_x(1,t) + 0.2u(1,t) = 0 \text{ for } t > 0$$

by means of the explicit and the implicit method using $h = 0.1$ and $h = 0.05$ for different time steps k. Derive the condition of the absolute stability for the explicit method in the case of the given boundary conditions. Compare your approximate solution to the actual solution $u(x,t)$ as $t \to \infty$, where $u_t = 0$. **[25 marks]**

2. Consider the boundary value problem for the trapezoidal region G shown in the diagram below:

$$u_{xx} + u_{yy} = -1 \quad \text{in } G$$
$$u = 1 \text{ on } AB; \ \frac{\partial u}{\partial n} = 0 \quad on \quad BC;$$
$$u = 0 \text{ on } CD \text{ and } \frac{\partial u}{\partial n} + 2u = 1 \text{ on } DA$$



Determine the difference equation on the basis of a five-point approximation scheme using $h = \frac{1}{4}$.

What are the difference equations for the mesh points, where the mesh size is $h = 1$? Is the system of difference equations symmetric? Set up the matrix form of the linear system of equations associated with this problem. **[25 marks]**

3. a)Classify the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x}\left(x(1-x)\frac{\partial u}{\partial t}\right) = (2 + x - x^2)\frac{\partial^2 u}{\partial t^2}.$$ **[5 marks]**

b) Use the method of characteristics to find the solution at the characteristic grid points R and T in the xt -plane (see diagram below) bounded by $x = 0.2$ and $x = 0.4$, for

[illegible]

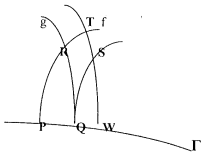
$$\frac{\partial^2 u}{\partial x^2} + xt\frac{\partial^2 u}{\partial x \partial t} = \frac{\partial^2 u}{\partial t^2}$$

with the initial conditions

$$u(x,0) = x \ , \ u_t(x,0) = 0 \ ,$$

and the boundary conditions

$$u(0,t) = 0 \ , \ u(1,t) = 0 \ .$$



Γ is a non-characteristic curve along which the initial values for $u, \frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial t}$ are known. The f characteristic through P at $x = 0.2$ intersects with the g characteristic through Q at $x = 0.4$.

[20 marks]

4. Consider the Initial/Boundary-Value (Parabolic) Problem given by:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} ; 0 < x < X, \ t > 0 \qquad \text{(i)}$$

Boundary conditions :

$$u(0,t) = u(X,t) = 0, \ t > 0 \qquad \text{(ii)}$$

Initial distribution :

$$u(x,0) = g(x); \ 0 \le x \le X \qquad \text{(iii)}$$

where $g(x)$ is a given continuous function of x.

The region $\Re$ and its boundary $\partial\Re$, consisting of the lines $x = 0, x = X$ and $t = 0$, are covered by a rectangular mesh of points with coordinates

$$(x_m, t_n) = (mh, \text{ln}); \ m = 0,.., N, N+1; n = 0,1,2,.....$$

Let the analytical solution of {(i),(ii),(iii)} at (x_m, t_n) be $u(x_m, t_n)$, the solution of an approximating difference scheme at (x_m, t_n) be denoted by u_m^n and the computed solution actually obtained(which will include round-off errors) be denoted by $\tilde{u}_m^n$.

(a) For this problem, described here, obtain the following:
  (i) The semi-discretised form.
  (ii) An explicit, four-point, two-time level, finite difference method for solving {(i),(ii),(iii)}.
  (iii) The weighting diagram for the four-point, two-time level, finite difference method
  (iv) An implicit six-point, two-time level finite difference method. **[16 marks]**

(b) Show that the local truncation error for an explicit, four-point, two-time level, finite difference method has principal part equal to

$$-\frac{1}{2} l \frac{\partial^2 u}{\partial t^2} - \frac{1}{12} h^2 \frac{\partial^4 u}{\partial x^4} .$$

[8 marks]

5. The weak form of the problem

$-\Delta[u(x,y)] = g(x,y);(x,y) \in \Omega; u(x,y) = 0,(x,y) \in \partial\Omega,$

where Ω is a polygonal domain with boundary $\partial\Omega$ and $g(x,y)$ is known, is such that

$u(x,y) \in \overset{o}{H}{}^1(\Omega)$ satisfies

$$a(u,v) = (g,v) \text{ for all } v \in \overset{o}{H}{}^1(\Omega). \qquad \text{(i)}$$

where $a(u,v) \equiv \int \nabla u . \nabla v dx$, $(g,v) = \int_\Omega gv dx$. Problem (i) is approximated by the Galerkin problem in which $U(x,y) \in S \subset \overset{o}{H}{}^1(\Omega)$ satisfies

$$a(U,V) = (g,V) \text{ for all } V \in S. \qquad \text{(ii)}$$

where S is the space of continuous piecewise linear functions defined over a partition of Ω into triangular elements Ω^e. The approximation U is calculated using a linear isoparametric finite element method in which the standard triangle T in the (ξ,η) plane is mapped in turn onto each element of the partition, with vertices $(x_i,y_i),(x_j,y_j),(x_k,y_k)$, and a linear approximation $\hat{U}(\xi,\eta)$ is used in T.

Show that $a(u,v)_{|\Omega^e} - (g,v)_{|\Omega^e}$ can be in the form

$$\int_T \left[\left(\frac{x_\mu^2 + y_\eta^2}{|J|}\right)\hat{u}_\xi \hat{v}_\xi - \left(\frac{x_\xi x_\eta + y_\xi y_\eta}{|J|}\right)(\hat{u}_\xi \hat{v}_\eta + \hat{u}_\eta \hat{v}_\xi) + \left(\frac{x_\xi^2 + y_\eta^2}{|J|}\right)\hat{u}_\eta \hat{v}_\eta\right] d\xi d\eta$$

$$- \int_T \hat{g}(\xi,\eta)\hat{v}(\xi,\eta)|J| d\xi d\eta,$$

where, under the transformation,

$u(x,y) \to \hat{u}(\xi,\eta), v(x,y) \to \hat{v}(\xi,\eta), g(x,y) \to \hat{g}(\xi,\eta)$

and J is the Jacobian of the transformation.

Hence derive the element stiffness matrix for Ω^e and, leaving each term in integral form, the element constant vector. Indicate how you would set up the global stiffness equations to solve the Galerkin problem.

[25 marks]