National University of Science and Technology
Department of Applied Mathematics

SMA 5273 OPERATIONS RESEARCH TECHNIQUES FOR MANAGEMENT

May/June 2004 **3 hours**

Answer **ALL** questions in section A and any **THREE** from section B

Section A (40 marks)

Answer ALL questions in this section

A1. Prove that the optimum solution of the linear programming model

Max $z = \underline{CX}$

s.t. $(\underline{A}, \underline{I})\underline{X} = \underline{b}$

and $\underline{X} \geq 0$

when finite, occurs at an extreme point of the solution space. **[10marks]**

A2 (a) State a sufficient condition for a stationary point $\underline{X}_o$ to be an extremum.

(b) A monopolist producing a single product has two types of customers. If q_1 units are produced for customer 1, then customer 1 is willing to pay a price of $70 - 4q_1$ dollars. If q_2 units are produced for customer 2, then customer 2 is willing to pay a price of $150 - 15q_2$ dollars. For $q > 0$, the cost of manufacturing q units is $100 + 15q$ dollars. How many units should the monopolist sell to each customer so that he maximizes profit? **[10marks]**

A3. A construction contractor estimates that the size of the workforce needed over the next 5 weeks to be 5, 7, 8, 4 and 6 workers respectively. Excess labour kept on the force will cost $300 per worker per week and new hiring in any week will incur a fixed cost of $400 plus $200 per worker per week. Determine an optimum plan for the contractor. **[12marks]**

A4. Suppose that commuter omnibuses from City Hall to NUST come randomly at 8 minute intervals. If the service were not random but at fixed intervals of 8 minutes, we expect the average waiting time per passenger to be 4 minutes. However, due to traffic jams the commuter omnibuses arrive in a random fashion, one every 8 minutes. Prove that the average waiting time is 8 minutes per passenger, i.e. double the amount for the case when arrivals were not random. **[8marks]**

Section B (60 marks)

Answer any **THREE** questions from this section

B5. (a) In an $(M/M/1):(FIFO/\infty/\infty)$ system, assume that the arrival and departure rates are λ_n and μ_n where n is the number of customers in the system at that time.

(i) Prove that the steady-state probabilities are given by

$$p_n = p_o \prod_{m=1}^{n} \frac{\lambda_{m-1}}{\mu_m}, n = 1,2,...$$

(ii) Let $\lambda_n = \lambda$ and $\mu_n = \mu$ for all n. Prove that in the steady-state

$L_s = \dfrac{\lambda}{\mu - \lambda}$ where L_s is the expected number of customers in the queueing system.

(b) A machine shop has six machines and two mechanics to repair faulty machines. Each machine breaks down in a Poisson fashion with a mean of 2 per hour. The repair time on each machine follows an exponential distribution with a mean of 20 minutes.

(i) Find the probability that the two mechanics are idle.

(ii) Find the expected number of idle mechanics.

(iii) What is the expected number of faulty machines waiting for repair? **[20marks]**

B6. Consider the following linearly constrained optimisation problem

Max	$f(x_1, x_2) = \ln(1 + x_1) + x_2$
s.t.	$2x_1 + x_2 \le 3$
and	$x_1, x_2 \ge 0$

where ln denotes natural logarithm.

(a) Use the KKT conditions to derive an optimal solution.

(b) Starting from the initial trial solution $(x_1, x_2) = (0,0)$, use one iteration of the Frank –Wolfe algorithm to obtain exactly the same solution you found in part (a) and then use a second iteration to verify that it is an optimal solution (because it is replicated) **[20 marks]**

B7. A highway construction firm rents certain of its specialized earth-moving equipment from a leasing company as needed during the various phases of a construction job.

Equipment has to be rented for full weeks and it can only be obtained at the beginning of a week and released at the end of a week. Assume that the beginning of a week coincides with the end of the preceding week. The rental cost per week is \$200. Each time a unit is rented there is a preparation and transport charge of \$120 by the leasing company. Each time a unit is returned to the leasing company, there is a servicing, cleaning and transport charge of \$150. No units will be required after 6 weeks. Therefore all remaining units are released at the beginning of week 7.The number of units required during any time period of 1 week cannot be predicted exactly, since the requirement depends on a variety of factors, such as the weather and soil conditions. If during any week, the construction firm has fewer units on hand than the number required (i.e. the company is short) it has to farm out the excess work to a local contractor at a cost of \$400 per unit short per week. The table below shows the average number short for various possible units rented during each week of the planning horizon. Fractions denote units required for less than a full week. This information implies that in no week will more than 4 units be required, since no work is farmed out if 4 units are rented.

Number of units rented during week n	**Average number short $R_n(z_n)$**					
	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$Z_n=0$	1.2	2.0	0.2	0.5	2.2	2.0
1	0.4	1.0	0	0	1.2	1.0
2	0	0.3	0	0	0.4	0.2
3	0	0	0	0	0.1	0
4	0	0	0	0	0	0

Given that the firm has 3 units on hand initially, what is the rental policy that minimizes the total cost incurred over the entire 6 weeks? **[20 marks]**

B8. Consider the 6-city travelling salesman problem for which the travelling costs c_{ij} are given in the table below. The salesman lives in city 1. What order of visiting the other cities will minimise the total distance travelled using

(a) the Nearest Neighbour Heuristic

(b) the Tour – building algorithm

	1	**2**	**3**	**4**	**5**	**6**
1	0	3	10	5	17	4
2	5	0	8	11	2	25
3	8	16	0	5	9	5
4	4	5	7	0	11	14
5	18	9	13	4	0	7
6	21	7	15	6	20	0

[20marks]

END OF EXAMINATION