NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 5282

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

SMA 5282: FINANCIAL MATHEMATICS

DECEMBER 2002

Time : 3 hours

Candidates should attempt **ALL** questions from Section A and **ANY THREE** questions from Section B.

SECTION A: [25 marks]

Answer **ALL** questions from this section.

A1. Let B_T be 1-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P), B_0 = 0$ $a.s$ P. Use Itó's formula to show that $M_t := e^{\frac{1}{2}t} \sin B_t$ is a martingale with respect to P. [5]

Find $E_P[M_T]$. [5]

A2. Consider the 1-period model

$$X_0 = (1, 1),\ X_T = (1, Y) \in \mathbb{R}^2$$

where Y is a normally distributed random variable with mean 2 and variance 1.

Show that there is no arbitrage for this market. [9]

A3. Consider a multi-period n-dimensional model $\{X_t\}_{t \in \mathcal{T}}$,

where $\mathcal{T} = \{0 = t_0, t_1, t_2, \ldots, t_n = T$.

Suppose X_t is a martingale with respect to a measure Q and with respect to a filtration $\{\mathcal{F}\}_{t \in \mathcal{T}}$, where the σ-algebra $\mathcal{F}_t$ represents the information available to the agent at time t.

Let $\theta_t = (\theta_t^{(1)}, \ldots, \theta_t^{(n)})$ be an $\mathcal{F}_t$- adapted, self financing portfolio. Then the fortune gained from time $t = 0$ to time $t = T$ is given by

$$V_T - V_0 = \sum_{s=0}^{t_n - 1} \theta_s.(X_{s+1} - X_s)$$

Prove that $E_P[V_T - V_0] = 0$. [6]

SECTION B: [75 marks]

Answer **ANY THREE** questions from this section. Each question carries 25 marks.

B4. (a) Find a self-financing portfolio $\theta_t = (\theta_t^{(1)}, \theta_t^{(2)})$ which hedges the T-claim $F(\omega) = \sin B_T(\omega)$ in the market $X_t = (1, B_t) \in \mathbb{R}^2; t \in [0, T]$ [15]

(b) Suppose we have a 1-period model with a 2 dimensional price process $X_t; t = 0, T$ such that $X_0 = (1,1), X_T = (1, Z)$ where Z is a random variable such that $P[Z = 0] = \frac{1}{4}, P[Z = 1] = \frac{1}{2}, P[Z = 2] = \frac{1}{4}$
Prove that this market is not complete. [10]

B5. Consider the Black-Scholes market
$X_t = (e^{\rho t}, Y_t) \in \mathbb{R}^2$
where
$dY_t = -Y_t dt + 4Y_t dB_t \in \mathbb{R}, Y_0 = 1.$
Find the price at time $t = 0$ of a European option with payoff
$F(\omega) = \exp(B_T(\omega))$ at the time of maturity $t = T$. [25]

B6. (a) Find two equivalent martingale measures Q_1, Q_2 for the price process
$X_t = (1, Y_t) \in \mathbb{R}^2$
where $dY_t = dB_t^{(1)} - 5dB_t^{(2)}$,
$(B_t^{(1)}, B_t^{(2)})$ being 2-dimensional Brownian motion. [8]

(b) Is the market arbitrage-free? Complete?. [2,2]
Find a T-claim $F(\omega), E[F^2] < \infty$ which is not attainable in this market. [13]

B7. Suppose a market $X_t = (1, Y_t^{(1)}, Y_t^{(2)}) \in \mathbb{R}^3$ is defined by

$$dY_t^{(1)} = dt + dB_t^{(1)} + dB_t^{(2)}; \ Y_0^{(1)} = 0$$
$$dY_t^{(1)} = 2dB_t^{(1)} + 2dB_t^{(2)}; \ Y_0^{(2)} = 0$$

(a) Show that there is no equivalent martingale measure for this market. [15]

(b) Does the system have an arbitrage? If so, find one. [10]

END OF QUESTION PAPER