NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY
FACULTY OF APPLIED SCIENCE
COMPUTER SCIENCE DEPARTMENT
DECEMBER EXAMINATIONS 2004

SUBJECT: Discrete Mathematics
CODE: SCS 5102

INSTRUCTIONS TO CANDIDATES
The question paper consists of seven (6) questions
Answer any 4 questions

QUESTION ONE

a) Define the following Terms:
   i) Theorem [2]
   ii) Deduction [2]

b) Using a the predicate logic prove the theorem:

$$(\forall x)[P(x) \wedge Q(x)] \rightarrow (\forall x)P(x) \wedge (\forall x)Q(x)$$ [8]

c) Use mathematical induction to prove that the statement is true for every positive integer n.

$$1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{n+1} n^2 = \frac{(-1)^{n+1} n(n+1)}{2}$$ [7]

d) Let X and Y be positive numbers, and prove that $X < Y$ if and only if $X^2 < Y^2$.
[6]

QUESTION TWO

a) Consider the set $A = \{1, 2, 3, 4, 6, 9\}$. Define a relation R on A by writing $(x, y) \in R$ if and only if $x - y$ is a multiple of 3.

i. Describe R as a subset of $A \times A$. [3]

ii. Show that R is an equivalence relation on A. [6]

iii. What are the equivalence classes of R? [6]

b) Solve the recurrence relation subject to the basis step by using the expand guess, and verify approach.

F (n)=1

F (n)=nF (n -1) [10]

QUESTION THREE

a) For each of the following sentences, write down the sentence in logical notation, negate the sentence, and say whether the sentence or its negation is true:

i. Given any integer, there is a larger integer. [2]

ii. There is an integer greater than all other integers. [2]

iii. Every even number is a sum of two odd numbers. [2]

iv. Every odd number is a sum of two even numbers. [2]

v. The distance between any two complex numbers is positive. [2]

b) Let A be a finite set with 7 elements, and let L be a finite language on A with 9 elements such that $\lambda \in L$.

i) How many elements does A^3 have? [4]

ii) Explain why L^2 has at most 73 elements. [4]

c) Show that if $a \in \mathbb{N}$ and $b \in \mathbb{Z}$. Then there exist unique $q, r \in \mathbb{Z}$ such that $b = aq + r$ and $0 \leq r < a$. 7]

QUESTION FOUR

a) It is well known that every multiple of 2 must end with the digit 0, 2, 4,6 or 8, and that every multiple of 5 must end with the digit 0 or 5.

Prove the equally well-known rule that a natural number is a multiple of 3 if and only if the sum of the digits is a multiple of 3 by taking the following steps.

Consider a k-digit natural number X, expressed as a string $x_1 x_2 \ldots. x_k$,

where the digits $x_1 x_2 \ldots. x_k \in \{0, 1, 2, \ldots, 9\}$.

i) Calculate the value of X in terms of the digits $x_1 x_2 \ldots. x_k$ [4]

ii) Calculate the difference between X and the sum of the digits. [4]

iii) Show that this difference is divisible by 3. [5]

iv) Complete the proof. [4]

b) Let x, y, m, n, a, b, c, d $\in \mathbb{Z}$ satisfy m = ax + by and n = cx + dy with ad - bc = ±1. Prove that (m, n) = (x, y). [8]

QUESTION FIVE

a) Let A = {a, b, c, d}

B = {a, b, e, g, h}

C = {b, d, e, g, h, m, n}

Verify:

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cup B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$ [10]

b) Justify each of the steps in the following proof sequence of:

$(P \to Q) \wedge [P \to (Q \to R)] \to (P \to R)$

i) $P \to Q$

ii) $P \to (Q \to R)$

iii) $[P \to (Q \to R)] \to [(P \to Q) \to (P \to R)]$

iv) $(P \to Q) \to (P \to R)$

v) $P \to R$ [15]

QUESTION FOUR

a) It is well known that every multiple of 2 must end with the digit 0, 2, 4,6 or 8, and that every multiple of 5 must end with the digit 0 or 5.

Prove the equally well-known rule that a natural number is a multiple of 3 if and only if the sum of the digits is a multiple of 3 by taking the following steps.

Consider a k-digit natural number x, expressed as a string $x_1 x_2 \dots x_k$, where the digits $x_1 x_2 \dots x_k \in \{0, 1, 2, \dots, 9\}$.

i) Calculate the value of x in terms of the digits $x_1 x_2 \dots x_k$ [4]

ii) Calculate the difference between x and the sum of the digits. [4]

iii) Show that this difference is divisible by 3. [5]

iv) Complete the proof. [4]

b) Let x, y, m, n, a, b, c, d $\in \mathbb{Z}$ satisfy m = ax + by and n = cx + dy with ad - bc = ±1. Prove that (m, n) = (x, y). [8]

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a) Let
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Verify:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cup B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$ [10]

b) Justify each of the steps in the following proof sequence of:

$$(P \to Q) \wedge [P \to (Q \to R)] \to (P \to R)$$

i) $P \to Q$

ii) $P \to (Q \to R)$

iii) $[P \to (Q \to R)] \to [(P \to Q) \to (P \to R)]$

iv) $(P \to Q) \to (P \to R)$

v) $P \to R$ [15]

QUESTION SIX

a) Suppose that T = O = {0, 1, 2, 3, 4, 5}.

i) Design a Finite state machine, which inserts the digit 0 at the beginning of any string beginning with 0,2 or 4, and which inserts the digit 1 at the beginning of any string beginning with 1,3 or 5. Describe your result in the form of a transition table. [8]

ii) Design a finite state machine, which replaces the first digit of any input string beginning with 0,2 or 4 by the digit 3. Describe your result in the form of a transition table. [8]

b) Describe the following search Algorithms. Exemplify where possible.

i) Depth-First Search [3]

ii) Breadth-First Search [3]

iii) Shortest Path Problem [3]

END OF QUESTION PAPER

GOOD LUCK!