NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

B.COMM (HONOURS) DEGREE IN ACTUARIAL SCIENCE

ACTUARIAL STATISTICS III – CIN 2211

APRIL/MAY 2003 SECOND SEMESTER EXAMINATION

INSTRUCTIONS TO CANDIDATES

1. Attempt all 11 questions, beginning your answer to each question on a separate sheet.
2. March allocations are shown in brackets

Additional Examination Material
Non-programmable electronic calculator
Tables of Actuarial Examinations (Actuarial Tables)

Question 1

(a) Given a measurable space(⃠, F), what is a probability measure defined on this space? **[1 mark]**

(b) Let (⃠, F) be a measurable space and let $P_1, P_2, ..., P_n$ be a collection of probability measures defined on F. If $a_1, a_2, ..., a_n$ is a collection of non-negative real numbers such that $\sum_{i=1}^{n} a_i = 1$, , prove that functions P^* defined on F by $P^*(E) = \sum_{i=1}^{n} a_i P_i(F)$ is a probability measure on F.

[3 marks]

(c) Let (⃠, F) be a measurable space and let $P_1, P_2, ...$ be a sequence of probability measures defined on F. Prove that the function P* defined on F by

$$P^*(E) = \sum_{n=1}^{\infty} \tfrac{1}{2^n} P_n(E)$$

is a probability measure on F. **[3 marks]**

(d) Let $\{K_i\}_{i.⁈I}$ be a family of ⑨ - algebras on ⃠, Prove that

$$\bigcap_{i \in I} K_i$$

is again a ⑨-algebra **[3 marks]**

[Total 10 marks]

Question 2

(a) (i) Define a linear congruential generator (LCG) explaining clearly how actual pseudo-random numbers are generated. [3 marks]

(ii) With the following choice of constants to be used in (i) above, generate 5 pseudo-random numbers from the U(0, 1) distribution: $a = 13$ $c = 39$ $m = 100$ $X_0 = 87$ (seed). [3 marks]

(b) (i) State any two methods that may be used in general to generate random variates from a specified distribution. [2 marks]

(ii) Explain in detail how you would simulate a variate from the Pareto distribution with parameters α, λ ($\alpha > 0, \lambda > 0$) using the inverse transform method. [3 marks]

(iii) What are the disadvantages of the inverse transform method in general? [1 mark]

(c) Choose one of the following options and give a full explanation with or without any mathematical expressions. [3 marks]

(i) Explain in detail how you would simulate the share price process with both positive or negative jumps over a time period and state a possible application for this simulation.

(ii) Explain in detail how you would simulate the cumulative claims process of an insurance company over a time period and state a possible application for this simulation.

[Total 15 marks]

Question 3

(a) Define a stopping time and give two examples of stopping times. [2 marks]

(b) In a coin tossing game, you either win or lose $1 depending on the outcome of the toss. If you start the game with $30 in your pocket and decide to play until you have $50 or $10 and X_n is the amount you have at step n, then the time you stop the fame is defined by

$$\tau = \min\{n : X_n = 10 \text{ or } 50\}$$

Prove that τ is a stopping time w.r.t the filtration

$$F_n = \sigma\{X_1, X_2, ..., X_n\}$$ [3 marks]

(c) Let $U = [a, b]$ $a < b$, $a, b \in R$

Let X_t be a Brownian motion with $X_0 = c$ and $a < c < b$
Define the filtration $M_t = \sigma\{X_s : 0 [s [t\}$ and a time

$$\tau = \inf\{t > o \; ; X\,t \notin U\}$$

Prove that τ is a stopping time w.r.t the filtration Mt [3 marks]

(d) Given a discount rate ❾ and a well behaved price process Xt and American call on Xt, with strike price K and expiring at time T from now. Give an expression of the price of the call price at time zero as an optimal stopping problem. [2 marks]

[Total 10 marks]

Question 4.

(a) (i) Let B_t be a Brownian motion with $B_0 = 0$ and define the region

$$D_a = \{ x \mid -a < x < a, x, a \in IR\} \; a = \text{where a is a constant.}$$

Prove that

$$P(B_t \in D_a) = 2\Phi\left(\frac{a}{\sqrt{t}}\right) - 1$$ [3 marks]

(iii) Let B_t be a Brownian motion of IR, $B_0 = 0$
Prove that:

$$E[B_t^{2k}] = \frac{(2k)!}{2^k k!} t^k \quad k \in IN$$ [4 marks]

[You may find the following duplication formular for a gamma function useful

$$\Gamma(n + \tfrac{1}{2}) = \sqrt{\pi} \, . \, \frac{\Gamma_{(2n)}}{2^{2n-1}\Gamma_{(n)}}$$

(iii) Define a diffusion and explain how it is related to the Brownian motion. [2 marks]

(iv) What is a mean reverting process and what can be modelled by it? [1 mark]

(b) (i) Why is a Geometric Brownian motion a better model of share prices rather than the Brownian motion itself? [2 marks]

(ii) Define a Levy process and explain how it differs from a Brownian motion. [2 marks]

(iii) Explain how you can model the contribution to the Levy process X_t of jumps that are in the range (a, b), and how you would also model the number of jumps of sizes in (a, b).

[4 marks]

(iv) State the stochastic differential equation of a Geometric Levy process. [1 mark]

(v) Why is a Geometric Levy process a better model of share prices as compared with the Geometric Brownian motion in some markets?

[1 mark]

[Total 20 marks]

Question 5

(a) (i) For a Markov chain, what is a stationary probability distribution? [2 marks]

(ii) What is an irreducible Markov chain? [1 mark]

(iii) What can be said about the stationary probability distribution of an irreducible Markov chain with a finite state space?

[1 mark]

(b) Each year a man trades his car for a new one. If he has an Alfa Romeo, he trades it for a BMW. If he has a BMW he trades it for a Corolla. However if he has a Corolla, he is just as likely to trade it for a new Alfa Romeo as to trade it for a BMW or a Corolla. In the year 2000, he bought his first car, which is a corolla.

Find the probability that he has a

(i) 2002 Corolla

(ii) 2002 Alfa Romeo

(iii) 2003 BMW

(iv) 2003 Corolla [9 marks]

(c) For the man in (b) above, how often will he have each of the three cars in the long run? Express as % of time. [4 marks]

[Total 17 marks]

Question 6

(a) (I) Let Xt be such that

$$dX_t = \mu dt + ⑨dBt$$

and let $f: \Re \to \Re$ be twice differentiable.

By considering $f(Xt)$, state Ito's lemma. [1 mark]

(ii) Let Xt = ct + αBt and note that

$dXt = cdt + \alpha dBt$

We define the process Yt by

$Yt = e^{Xt}$

Prove that

$dY_t = (c + \tfrac{1}{2}\alpha^2)\, Y_t d_t + \alpha Y_t dB_t$ [3 marks]

(iii) Use I to 's lemma to prove that

$$\int_0^t B_s^2 dB_s = \tfrac{1}{3}B_t^3 - \int_0^t B_s ds$$ [3 marks]

(c) Find the solutions of the following Stochastic differential equations.

(i) $dX_t = X_t d_t + dB_t$ [consider $d(e^{-t}X_t)$] [2 marks]

(ii) $dX_t = (m - X_t)d_t + \sigma dB_t$ [mean reverting Ornstein-Uhlenbeck process.] [2 marks]

(c) (i) What is the distribution of

$\int_0^t f(s)dBs$ and what is

$E[\int_0^t f(s)dBs]$ and var $[\int_0^t f(s)dBs]$ [2 marks]

(ii) Using your results from c(i) above find

$E[X_t]$ and var $[X_t]$ for the process in b(ii) above.

[3 marks]

(iii) Given an interpretation of $E[X_t]$ [1 marks]

[Total 17 marks]

Question 7

(a) Obtain the Auto Correlation Function (ACF) of the MA(2) process

$Z_t = a_t - \tfrac{3}{4}a_{t-1} + \tfrac{1}{8}a_{t-2}$

in terms of $(a_t) = \sigma_a^2 = \text{var}(a_{t-k})$ for k = 1, 2.

Where at a_t is a white noise process. [4 marks]

(b) State conditions that must be satisfied to ensure the stationarity and invertibility of AR(p) and MA(q) process respectively.

Are the following processes (i) stationary (ii) invertible?

(bi) $Z_t = \tfrac{1}{2}Z_{t-1} + a_t$

(bii) $Z_t = a_t - a_{t-1} + \tfrac{1}{2}a_{t-2}$

(biii) $Z_t = \frac{1}{2} Z_{t-1} + a_t + 2a_{t-1}$ [5 marks]

(c) A Markov process $Z_t + pZ_{t-1} = a_t$ $|p| < 1$ cannot be observed directly but the process $Y_t = Z_t + b_t$ is observed where a_t and b_t are uncorrelated white noise processes with variances σ_a^2, σ_b^2 respectively.

Show that the autocovaniance function of Y_t is given by

$$\gamma(_0) = \frac{\sigma_a^2 + (1-p^2)\sigma_b^2}{1-p^2}$$

$$\gamma(k) = \frac{(-p)^k \sigma_a^2}{1-p^2} \qquad k \geq 1$$

[6 marks]

(d) Given a time series of economic importance and an appropriate statistical package, explain the various steps necessary to identify, estimate and check a satisfactory model for the data.

[4 marks]

[Total 19 marks]

Question 8

(a) (i) What is a filtration defined on a measurable space (Ω, F)?

(ii) When is a stochastic process $\{X_t\}_t \geq 0$ on (Ω, F, P) called a martingale w.r.t filtration $\{M_t\}_{t\geq 0}$?

(iii) Why is the study of martingale processes important in "modern" finance theory?

[6 marks]

(b) Prove that the following processes are martingales w.r.t the filtration $F_t = \sigma\{B_s; 0 \leq s \leq t\}$

(i) $B_t^2 - t$

(ii) $e^{\lambda Bt - \frac{1}{2}\lambda^2 t}$

(iii) σB_t

[6 marks]

(c) (i) State the continuous time optional stopping theorem.

(ii) The price of a share follows a geometric Brownian motion

$S_t = S_0 e^{\sigma B_t + \mu_t}$ with $\mu > 0$ $B_o = o$

It is currently valued at S_o and will be sold when its price reaches $L > S_o$. How long would you expect this to take?

[5 marks]

[Total 17 marks]

END OF EXAMINATION PAPER