NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

BACHELOR OF COMMERCE (HONOURS) DEGREE

TOPICS IN APPLIED MATHEMATICS FOR ACTUARIAL SCIENCE – CIN 2215

APRIL/MAY 2003 SECOND SEMESTER EXAMINATION

TIME : 3 HOURS

INSTRUCTIONS TO CANDIDATES

1. Candidates should attempt ALL six question.
2. Start your answer to each new question on a separate sheet.

Question 1:

Consider the following model of a certain commodity market. It is assumed that the price P, supply S and demand D are functions of time and that the rate of change of the price is proportional to the difference between the demand and the supply.

$$\frac{dp}{dt} = k(D - S) \qquad k > 0$$

(a) If it is assumed that

$D = c - dP$ and $S = a + bP$ with $0 < P < {}^{c}/_{d}$ where a, b, c and d are constraints.

(i) Comment on the forms of D and S as models of demand and supply respectively. **[2 marks]**

(ii) Determine P(t) **[5 marks]**

(iii) Analyze the behaviour of P(t) as t increases **[3 marks]**

(b) If it is assumed instead that

$D = c - d.P$ and $S = a + b.P + q \sin \omega t$

(i) Comment on the forms of D and S in (b). **[2 marks]**

(ii) Determine P(t) **[7 marks]**

(iii) Analyze the behaviour of P(t) as t increases **[5 marks]**

[For part (b) you may find the following results useful

$$\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2}\left[a \sin bx - b \cos bx\right]$$

cos(A + B) = cos A cos B – sin A sin B]

[Total 24 marks]

Question 2:

Let X(t) denote the size of a population at time (t).
Let r be the net growth rate growth rate, given as a per capita rate. The rate of change of the population is given by

$$\frac{dx}{dt} = r(x,t).X$$

(i) Explain the reasoning under a logistic model that assumes that

$$r(x,t) = r_0(1-\frac{x}{k}) \qquad r,\ k > 0$$ **[2 marks]**

(ii) For the logistic model

$$\frac{dx}{dt} = r_0(1-\frac{x}{k})x$$

Find a solution of this differential equation given that at t = 0 X = X_0
[9 marks]

(iii) Show that if $x_0 > 0$ then X(t) approaches k as t increases. **[2 marks]**

(iv) By considering cases $X_0 > k$ and $X_0 < k$ separately, draw the graphs of X(t) as t increases indicating clearly the carrying capacity k of the population. **[4 marks]**

[Total: 17 marks]

Question 3:

(a) The fibonacci numbers a_0, a_1, a_2, ... are defined by the relation

$a_{n+2} = a_{n+1} + a_n$ where $a_0 = 0$ and $a_1 = 1$.

(i) Find the first eight fibonacci numbers. **[2 marks]**

(ii) Where are these numbers used in finance and what were they originally created for? **[2 marks]**

(iii) Find a formula for a_n. **[9 marks]**

(b) Solve the following difference equations:

(i) $a_{n+2} - 5a_{n+1} + 6a_n = 2n + 2$ if $a_0 = 1$ $a_1 = 1$ **[11 marks]**

(ii) $a_{n+3} - 2a_{n+2} - a_{n+1} + 2a_n = 0$ if $a_0 = 0$ $a_1 = 1$ $a_2 = 1$
Hence find a_{10} **[9 marks]**

(iii) $a_{n+2} - 4a_{n+1} - 4a_n = 0$ if $a_0 = 1$ $a_0 = 4$ **[7 marks]**

[Total 40 marks]

Question 4:

(a) Let A be a non-singular matrix with distinct eigenvalues. Let B be a vector of corresponding eigenvectors of A

(i) Prove that $[B^{-1}AB]^n = B^{-1}A^nB$ **[7 marks]**

(ii) If A = $\begin{pmatrix} 17 & -6 \\ 35 & -12 \end{pmatrix}$

Find A^n for n = 1, 2, 3, ... **[16 marks]**

(iii) Use the above result to find A^5 **[2 marks]**

(b)

(i) Give an expression of the inverse of a matrix A in terms of its cofactors. **[2 marks]**

(ii) Hence or otherwise find the inverse of the following matrix

$$\begin{pmatrix} 2 & 3 & 2 \\ 1 & 2 & -3 \\ 3 & 4 & 1 \end{pmatrix}$$

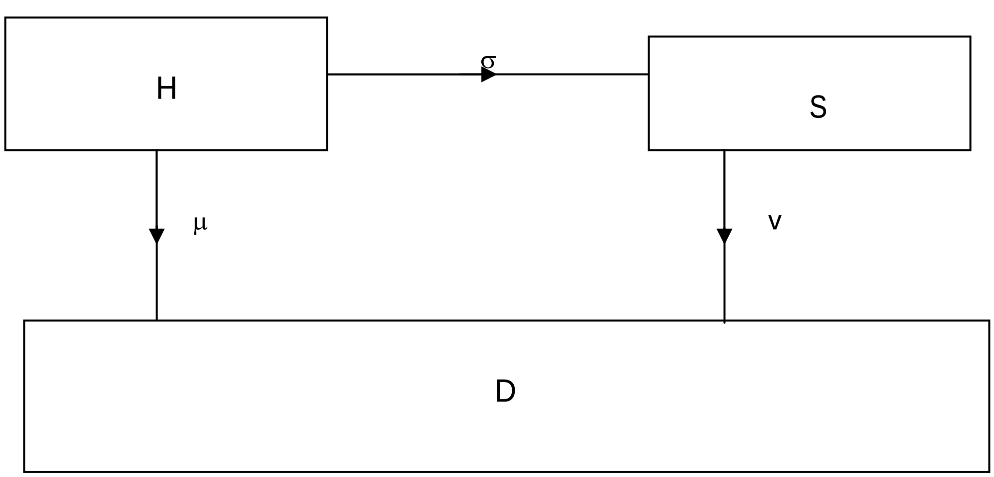
[8 marks]

(iii) Solve the system of equations

2x + 3y + 2z = 9
x + 2y – 3z = 14
3x + 4y + z = 16

[7 marks]
[Total 42 Marks]

<u>Question 5:</u>



The above is a multiple state model of a terminal disease. Once you get into state S you can only move to state D. Transitions from the healthy state can be to either state S or D. It can be shown that the transition probabilities satisfy the following differential equations. [You are not required to derive these]. Assume σ, μ , v are constants.

From H State:

$$\frac{d\,{}_tP_x^{HH}}{dt} = -(\sigma+\mu)\,{}_tP_x^{HH} \qquad \text{with } {}_0P_x^{HH} = 1 \qquad \text{(e1)}$$

$$\frac{d\,{}_tP_x^{HS}}{dt} = {}_tP_x^{HH}.\sigma - v.{}_tP_x^{HS} \qquad \text{with } {}_0P_x^{HS} = 0 \qquad \text{(e2)}$$

$$\frac{d\,{}_tP_x^{HD}}{dt} = {}_tP_x^{HH}.\mu + {}_tP_x^{HS}.v \qquad \text{with } {}_0P_x^{HD} = 0 \qquad \text{(e3)}$$

From S state:

$$\frac{d}{dt}\,{}_tP_x^{SS} = v.{}_tP_x^{SS} \qquad \text{with } {}_0P_x^{SS} = 1 \qquad \text{(e4)}$$

$$\frac{d\,{}_tP_x^{SD}}{d_t} = v.{}_tP_x^{SS} \qquad \text{with } {}_oP_x^{SD} = 0 \qquad \text{(e5)}$$

(i) Solve equation (e1) with the given boundary conditions. **[7 marks]**

(ii) What type of differential equation is (e2)? Solve (e2) taking into account the given initial conditions and any other useful result. **[10 marks]**

(iii) Either by solving completely, or giving a detailed explanation, how would you solve (e3)? A solution or an explanation is sufficient. **[7 marks]**

(iv) Solve differential equation (e4) **[7 marks]**

(v) Solve differential equation (e5) using any other results you might need. **[7 marks]**

(vi) In solving (e1) and (e4), for the solution to exist, what did we assume about ${}_tP_x^{HH}$ and ${}_tP_x^{SS}$ as functions of t? **[2 marks]**

[Total 10 marks]

<u>Question 6</u>

(a) Compute the following determints

$$\begin{vmatrix} 6 & -5 & 8 & 4 \\ 9 & 7 & 5 & 2 \\ 7 & 5 & 3 & 7 \\ -4 & 8 & -8 & 3 \end{vmatrix}$$

[10 marks]

(c) Find the eigenvalues and corresponding eigenvectors of the following matrix:

$$A = \begin{pmatrix} -2 & -9 & 5 \\ -5 & -10 & 7 \\ -9 & -21 & 14 \end{pmatrix}$$

[17 marks]

(c) Find matrices which transform matrix in (b) into diagonal form. **[10 marks]**

[Total 37 marks]

END OF EXAMINATION PAPER