NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

FACULTY OF COMMERCE

DEPARTMENT OF INSURANCE AND ACTUARIAL SCIENCE

B.COMM (HONOURS) DEGREE IN ACTUARIAL SCIENCE

TOPICS IN APPLIED MATHEMATICS - CIN 2215

AUGUST 2009 - SECOND SEMESTER EXAMINATIONS

DURATION: 3 HOURS

Instructions to Candidates

1. **Attempt all SIX (6) Questions**
2. **To Obtain Full Marks Show ALL appropriate steps to your answers**

Requirements

1. **Actuarial Tables (2002) Edition**
2. **Non-programmable Scientific calculator**

P1.

(i) Apply the Laplace expansion to reduce A down to co-factors of order 2 using column 3, given that:

$$A = \begin{bmatrix} 5 & 4 & 2 & 1 \\ 2 & 3 & 1 & -2 \\ -5 & -7 & -3 & 9 \\ 1 & 2 & -1 & 4 \end{bmatrix}$$

[4 marks]

(ii) Apply Cramer's rule to solve the following set of simultaneous equations for all values of ***p***:

$$\begin{aligned} px + y &= 1 \\ x - y + z &= 0 \\ 2y - z &= 3 \end{aligned}$$

[6 marks]

[Total 10 Marks]

P2. A Markov Chain has the transition matrix

$$T = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

(i) Show that T has just two distinct non-trivial eigenvalues. **[5 marks]**

(ii) Find the three eigenvectors of T denoted by $\mathbf{r}_1, \mathbf{r}_2$, and $\mathbf{r}_3$. **[5 marks]**

(iii) Find the Jordan decomposition matrix **J**, a matrix **C** and its inverse **C**$^{-}$1 such that T = $\mathbf{CJC}^{-1}$ given that:

$$\mathbf{J} = \begin{bmatrix} \lambda_2 & 1 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$, where λ_2 is a non unit eigenvector. **[5 marks]**

(iv) Input a formula for $\mathbf{J}^n$ and find T^n and its limit as $n \to \infty$, interpret your result.

[5 marks]

[Total 20 Marks]

P3.

(a) Find the solutions of the following difference equations:

(i) $u_{k+1} - 3u_k + u_{k-1} + u_{k-2}$ **[4 marks]**

(ii) $u_{k+2} - 5u_{k+1} + 6u_k = 2k + 2$, if $u_0 = 0,\ u_1 = 1$ **[5 marks]**

(b) The probability of a success in a single trial is $\frac{1}{3}$, if u_n is the probability that there are no two consecutive successes in ***n*** trials;

(i) Show that u_n satisfies: $u_{n+1} = \frac{2}{3}u_n + \frac{2}{9}u_{n-1}$ **[3 marks]**

(ii) What are the values of u_1 and u_2? **[3 marks]**

(iii) Hence show that: $u_n = \dfrac{1}{6}\left[\left(3 + 2\sqrt{3}\right)\left(\dfrac{1+\sqrt{3}}{3}\right) + \left(3 - 2\sqrt{3}\right)\left(\dfrac{1-\sqrt{3}}{3}\right)\right]$ **[4 marks]**

[Total 19 Marks]

P4.

(a) The logistic growth model of population at time **t, X(t)** is given by the following differential equation:

$$\frac{dX}{dt} = r_0\left(1 - \frac{X}{k}\right)X,\ X(0) = X_0,$$ where r_0 and k are positive constants

Show that the solution of the above differential equation is given by:

$$X(t) = \frac{k}{1 - \left(1 - \frac{k}{X_0}\right)e^{-r_0 t}}$$ **[6 marks]**

(b) Solve the following differential equation:

$$y^{11} + y^{1} + y = \sin x + 3\cos x$$ **[7 marks]**

[Total 13 Marks]

P5.

In a birth and death process with immigration the birth and death rates are given by $\lambda_n = n\lambda + \alpha$ and $\mu_n = n\mu$,where α represents a constant immigration rate.

(i) Show that the probability generating function G(s,t) satisfies:

$$\frac{\partial G(s,t)}{\partial t} = (\lambda s - \mu)(s-1)\frac{\partial G(s,t)}{\partial s} + \alpha(s-1)G(s,t)$$ **[5 marks]**

(ii) Show that, if $G(s,t) = (\mu - \lambda s)^{-\alpha/\lambda} S(s,t)$. Then $S(s,t)$ satisfies

$$\frac{\partial S(s,t)}{\partial t} = (\lambda s - \mu)(s-1)\frac{\partial S(s,t)}{\partial s}$$ **[3 marks]**

(iii) Assuming a modified initial condition for S(s,t) of an initial population size of n_0. Solve the partial differential equation for S(s,t) using a suitable change of variable and confirm that

$$G(s,t) = \frac{(\mu-\lambda)^{\alpha/\lambda}[(\mu-\lambda s) - \mu(1-s)e^{(\lambda-\mu)t}]^{n_0}}{[(\mu-\lambda s) - \lambda((1-s)e^{(\lambda-\mu)t}]^{n_0+(\alpha/\lambda)}}$$ **[6 marks]**

(iv)
(a) Find $p_0(t)$, the probability that the population is zero at time t. **[3 marks]**
(b) Why is there no extinction in this case? **[1 mark]**
(c) Hence show that $\lim_{t\to\infty} p_0(t) = \left(\frac{\mu-\lambda}{\mu}\right)^{\alpha/\lambda}$, if $\lambda < \mu$ **[1 mark]**
(d) What is the limit if $\lambda > \mu$? **[2 marks]**

(v) The long term behaviour of the process for $\lambda < \mu$ can be investigated by looking at the limit of the probability generating function as $t \to \infty$.

(a) Show that $\lim_{t\to\infty} G(s,t) = \left(\frac{\mu-\lambda}{\mu-\lambda s}\right)^{\alpha/\lambda}$ and interpret this result **[2 marks]**
(b) What is the long-term mean population? **[2 marks]**
[Total 25 Marks]

P5. Write short notes on the following steps in VBA programming in Excel which will display a 'Welcome' message box:

1. Planning and Design **[2 marks]**
2. Designing the interface **[3 marks]**
3. Setting the properties **[2 marks]**
4. Writing the code **[3 marks]**
5. Running the Application **[3 marks]**

[Total 13 Marks]

*****END OF EXAMINATION*****