NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

CIN 4111

FACULTY OF COMMERCE

DEPARTMENT OF ACTUARIAL SCIENCE

CIN 4111 Actuarial Statistics III/Stochastic Modelling

December 2005

Time : 3 hours

Attempt **All 7** questions.

A1. (a) State the axioms that make $(\Omega, \mathcal{F}, P)$ a probability space. [3]

(b) Let $(\Omega, \mathcal{F})$ be a measurable space and let$P_1, P_2, P_3, ..., P_n$ be a collection of probability measures defined on $\mathcal{F}$. If $a_1, a_2, a_3, ..., a_n$ is a collection of non-negative real numbers such that $\sum_{i=1}^{n} a_i = 1$ Prove that the function P^* defined on $\mathcal{F}$ by

$$P^*(E) = \sum_{i=1}^{n} a_i P_i(E)$$

is a probability measure on $\mathcal{F}$ [4]

(c) Let $\mathcal{H} = \{H_i\}_{i \in I}$ be a family of σ-algebras on Ω.
Prove that

$$H = \bigcap_{i \in I} \mathcal{H}$$

is again a σ-algebra. [5]

A2. Solve the following $stochastic\ differential\ equation (SDE)$ (B_t denotes a 1-dimensional Brownian motion) which can be used as model of long-term interest rates.

(a) $dX_t = a(\mu - (X_t - 1))dt + \sigma dB_t$ where $a, \mu, (\sigma > 0)$ are real constants and $B_t \in \mathbb{R}$ and given that $X(0) = X_0$. [3]

(b) Find $\mathbf{E}[X_t]$ and $Var[X_t]$ for X_t. [4]

(c) Give an interpretation of the expressions found in (b) and the merits of the SDE in (a) as a model. [2]

A3. (a)

(i) Let

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

be a stochastic matrix and let $\mathbf{u} = (u_1, u_2, u_3)$ be a probability vector. Show that the product $\mathbf{u}A$ is also a probability vector. [3]

(ii) What is an irreducible Markov chain? [2]

(iii) Let $\Pi_j, j \in S$,where S is the state space, be the probablity dustribution for a Markov chain with transion matrix P. State that conditions that should hold for Π_j to be the stationary probability distribution for the Markov chain. [4]

(b) A rating agency has three possing rating grades A, B and C (C is the worst grade). If a bank is rated A in a period, it is equaly likely to be rated A or C, but not B in the next period. If it is rated B in a period,it will definately be rated a B in the next period. If its in the C grade in this period, it is twice as likely to be rated a B as it is likely to be rated an A or a C in the next period.
Given that at the beginning of a certain period it is equaly likely to be rated A and B but its certain that its rated C.

(i) Find the transition matrix P of the rating process. [2]

(ii) Find the probability that the bank will be in grade C just after one period. [3]

(iii) Find the probability that the bank will remain in each of the rating classes after two periods, i.e $P^{(2)}_{AA}, P^{(2)}_{BB}$ and $P^{(2)}_{CC}$. [3]

(iv) In the long run, what is the probability distribution of the rating grades? [4]

A4. (a) Explain in detail how you would use simulation to price a European option driven by a geometric Brownian motion(GBM). [5]

(b) What is Cholesky factorization? [3]

(c) Give a detailed algorithm of how to simulate a variate from a discrete random variate. [4]

(d) Using the algorithm above,describe how you would simulate a variate from a **Binomial(3,0.7)**. [5]

A5. (a) Obtain the mean and the autocovariance function for each of the following processes. In each case , determine if the process is stationary (a_t is a zero mean white noise).

(i) $Z_t = b + ta_t$, b is a constant. [3]

(ii) $W_t = Z_t - Z_{t-1}$, where Z_t is as in (i). [3]

(iii) $Z_t = a_t a_{t-1}$. [3]

(b) (i) Show that the infinite order MA process Z_t defined by

$$Z_t = a_t + C(a_{t-1} + a_{t-2} + ...)$$

where C is a constant is non-stationary. [2]

(ii) Show that the first differences Y_t defined by

$$Y_t = Z_t - Z_{t-1}$$

is a first order MA process and is stationary, where Z_t is as in (b)(i) above. [3]

(iii) Find the autocorrelation function of Y_t. [3]

(c) (i) An autoregressive process X_t is defined by $X_t = aX_{t-1} + Y_t$, where the sequence Y_t is also defined autoregressively by $Y_t = aY_{t-1} + Z_t$, where b is not equal to a. Z_t is a sequence of uncorrelated random variables having mean zero and variance σ^2. Show that X_t may be alternatively expressed as a moving average $X_t = \sum_{r=0}^{\infty} d_r Z_{t-r}$ where $d_r = \frac{a^{r+1} - b^{r+1}}{a-b}$ and show that the autocorrelation function (ACF) of X_t is given by;

$$\rho_r = \frac{a^{r+1}(1-b^2) - b^{r+1}(1-a^2)}{(1+ab)(a-b)}$$

[5]

A6. (a) What is a martingale, $\{X_t\}_{t\geq 0}$ *with respect to* a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, P)$? [2]

(b) Prove that the following processes are martingales $w.r.t$ their natural filtrations.

(i) The compensated Poisson process

$$\tilde{N}(t) = N(t) - \lambda t$$

where $\{N(t); t \geq 0\}$ is a Poisson process with rate λ. [2]

(ii) $e^{\lambda B_t - \frac{1}{2}\lambda^2 t}$ where B_t is a Brownian motion on $\mathbb{R}$ and $\lambda \geq 0$ is a constant. [3]

(c) Define a *stopping time* with respect to a filtration? [2]

(d) State the Optional Sampling theorem. [2]

(e) (i) What is a Brownian motion, B_t defined on $\mathbb{R}$? [4]

(ii) Let B_t be Brownian motion and fix $t_0 \geq 0$. Prove that

$$\tilde{B}_t := B_{t_0+t} - B_{t_0};\ t \geq 0$$

is a Brownian motion. [5]

END OF QUESTION PAPER