NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

B. COMM (HONOURS) DEGREE IN ACTUARIAL SCIENCE

INTRODUCTION TO DERIVATIVES – CIN 4215

MAY/JUNE 2005 SECOND SEMESTER SUPPLEMENTARY EXAMINATION

DURATION : 3 HOURS

INSTRUCTIONS TO CANDIDATES

1. Answer all questions
2. Requirements – Scientific calculator
   Statistical Tables

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1. A stock price is currently \$40. Over each of the next two three-month periods it is expected to go up by 10% or down by 10%. The risk free interest rate is 12% per annum with continuous compounding.

   (a) What is the value of a six-month European put option with a strike price of \$42. **[8 marks]**

   (b) Under what circumstances would you exercise an American option early? **[3 marks]**

   (c) Is it in general possible to use the risk neutral valuation formula to value an American option. Justify your answer. **[3 marks]**

[Total: 14 marks]

2(a) Let π_t denote the value at time *t* of an option with payoff $X = \Psi(S_T)$ at time T, where S_t is assumed to follow the Black-Scholes model. Using arbitrage arguments, this value should be equal to the value V_t^h of its self-financing replicating portfolio h = (x,y) at all times *t* up to and including maturity. Show that under the risk-neutral measure, the rate of return from investing in the option is equal to the risk-free rate. **[6 marks]**

[Hint : In other words, prove that $\frac{d\pi_t}{\pi_t} = rdt + g_t d\tilde{W}_t$ for some stochastic process g_t]

(b) Hence show that the discounted option price $Z_t = B_t^{-1}\pi_t$ is a Q-martingale. **[4 marks]**

[Total:10 marks]

3. The current price of a stock is \$5. In three months time the price will be \$6 with probability 0.6 or \$4.50 with probability 0.4. The continuously compound risk-free rate of interest is 10% per annum. A call option on the stock has term three months and strike price \$5.50.

(a) Let t = 0 represent today and t = 1 represent three months from now. If $B_0 = 1$, what is the value B_1 of the bond after three months? Deduce the appropriate value of r to use in the binomial model. **[3 marks]**

(b) Derive from first principles an expression for the one-step weights q_u, q_d of an equivalent martingale measure Q. Hence find the value of q_u and q_d and check that the discounted stock price process is a Q-martingale. **[6 marks]**

(c) Calculate the arbitrage-free price of this option by:
(i) finding the replicating portfolio **[3 marks]**
(ii) using the risk-neutral pricing formula. **[2 marks]**

(d) Now suppose the call option above has a current price of \$0.1917. Identify an arbitrage opportunity available and clearly show the strategy that arbitrageurs could use to make free cash (if any) and how much? **[6 marks]**

[Total: 20 marks]

4. Suppose that the prices of a riskless bond B_t, and a risky asset S_t evolve according to the stochastic differential equations (SDEs)

$$dB_t = rB_t d_t$$
$$dS_t = \mu S_t d_t + \sigma S_t dW_t$$

Where r, μ and σ are positive constants, and W is a standard Brownian motion under the "real world" measure P.

(a) Which theorem can you use to find a measure Q, equivalent to P, such that

$$\tilde{W}_t = W_t + \left(\frac{\mu - r}{\sigma}\right)t$$

is a Brownian motion under Q?

Write down and simplify an expression for the Radon-Nikodym derivative $\frac{dQ}{dP}$. **[6 marks]**

(b) Using Itô's Lemma, determine an expression for $d(e^{-rT}S_t)$ and hence show that $e^{-rt}S_t$ is a Q-martingale. **[4 marks]**

[Total: 10 marks]

5. Identify the factors affecting the Black-Scholes price of a European call option and for each factor briefly discuss what happens when the variable (factor) in question changes slightly (reasons resulting from either the mechanics of the options or the Black-Scholes formula are acceptable). **[8 marks]**

6. (a) State the put-call parity relation **[2 marks]**

(b) Using the relation in (a) derive the Black-Scholes formula for the price of a European put option with strike price K and time to expiry T. **[6 marks]**

{Hint : You can use the fact that (1 – N[d]) = N[-d]}.

(c) A stock price is currently \$40 and its volatility is estimated at 20%. The risk-free rate of interest is r = 0.12. Find the value of a six-month European put option with strike price \$42 under the Black-Scholes model. **[4 marks]**

(d) Derive a formula for the time 0 value C_0 of a European call option with payoff

$$X = \max(S_t - K, 0)$$

Which depends on the strike price K, the value of the stock S_0 at time 0, the risk-free rate of interest r, the volatility σ of the stock and the time to maturity T.

Hint : The density function of an N[0,1] random variable is given by

$$\Phi(Z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$ **[9 marks]**

[Total:21 marks]

7. The Black-Scholes formula for the price at time t of a European call option with strike price K and maturity T (on a non-dividend paying stock), where S is the stock price at time t is

$C(t,S) = SN[d_1] - e^{-r(T-t)} KN[d_2]$,

where

$$d_1 = \frac{In(S/k) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$$

$$d_2 = \frac{In(S/k) + (r - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$$

and where

$$N[x] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\,dt$$

(a) What is $N^1[x]$? **[3 marks]**

(b) Calculate $\frac{\partial d_1}{\partial S}$ and $\frac{\partial d_2}{\partial S}$ **[6 marks]**

(c) Hence calculate $\frac{\partial C}{\partial S}$ **[4 marks]**

(d) In order to hedge a European call, how many units of the underlying stock should be held? **[4 marks]**

[Total: 17 marks]

END OF EXAMINATION PAPER!!