NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

B. COMM (HONOURS) DEGREE IN ACTUARIAL SCIENCE

INTRODUCTION TO DERIVATIVES – CIN 4215

JULY 2006 SUPPLEMENTARY EXAMINATION

DURATION : 32 HOURS

INSTRUCTIONS TO CANDIDATES

1. Answer all questions
2. Requirements – Scientific calculator
   Statistical Tables

QUESTION 1

Consider the following one-step model of bond (B) and stock (S):

$B_0 = 1,$ $\qquad B_1 = e^{\,r}$

$S_0 = s,$ $\qquad S_1 = \begin{cases} su, \\ sd, \end{cases}$

where, r, s, u, d and $p_d = 1 - p_u$ are constants.

(a) For a portfolio $h = (x, y)$, in which x(respectively y) represents the number of units of bond (respectively stock) held from time 0 to time 1, define

(i) the concept of a value process, **[2 marks]**
(ii) an arbitrage portfolio. **[2 marks]**

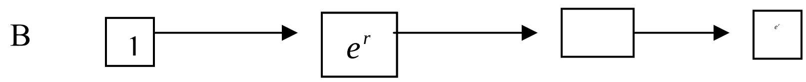
(b) Without a proof, state the conditions under which this model has arbitrage opportunities, and illustrate this with an explicit description of an arbitrage portfolio in a particular case. **[6 marks]**

(c) Derive from first principles an expression for the one-step weights q_u, q_d of an equivalent martingale measure. Comment briefly on the relationship; between the equivalent martingale weights and your answer to part (b).

[6 marks]
[Total 16 marks]

QUESTION 2

The diagram below show a model of a bond B and a non-dividend paying stick S . The probability that the stock follows a certain path is given by a probability measure P.



$t = 0$ $t = 1$ $t = 2$ $t = 3$

Note that for each step, the multiplicative factors for the stock price are $u = \frac{7}{6}$ and $d = \frac{5}{6}$
Suppose that $r = -$ In 0.9. Using risk neutral valuation, or other quicker methods, price (at time 0) the following options:

(a) A European call option with strike price 190 and expiry T = 2. **[3 marks]**
(b) An American call option with strike price 190 and expiry T = 2. **[3 marks]**
(c) A European put option with strike price 190 and expiry T = 2. **[3 marks]**
(d) An American put option with strike price 215 and expiry T = 3. **[4 marks]**

[Total 13 marks]

QUESTION 3

(a) Show that $S_t^{-\frac{2r}{6^2}}$ could be a price of a traded security. **[9 marks]**

(b) Let W_t be a standard Brownian motion, and let u, o be constants. By evaluating the stochastic differential d(In S_t, show that the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

has the unique solution

$$S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma Wt}$$

[Hint: Remenber to verify that S_t solves the SDE. You may assume that S_0 is a constant] **[8 marks]**

[Total 17 marks]

QUESTION 4

Suppose that the price of a riskless asset B_t, and a risky asset s_t evolve according to the SDEs

$$dB_t = rB_t dt$$
$$dS_t = \mu S_t dt + \sigma S_t dW_{t,}$$

where $r, \mu, and \sigma$ are positive constants, and W is a standard Brownian motion under the 'real world' measure .

(a) Which theorem can you use to find a measure Q, equivalent to P, such that

$$\tilde{W}_t = W_t + \left(\frac{\mu - r}{\sigma}\right)t$$

is a Brownian motion under Q? Write down and simplify an expression for the Radon-Nikodym derivative ……. **[5 marks]**

(b) Using It Lemma, determine an expression for d(d), and hence show that is a Q-martingale **[4 marks]**

[Total marks 9]

QUESTION 5

The price of a tradeable non-dividend paying asset St is given by

Where … is a positive constant, r represents the constant risk-free rate of interest, and …. Is a standard Brwonian motion under the risk neutral measure Q.

(a) (i) Calculate the expected value ……….. [Hint: Using the properties of the log-normal distribution, given a random variable Z ……..N(0,1) unde3r a probability measure Q we have ……………….. **[3 marks]**

(ii) By showing that

calculate the variance Var **[5 marks]**

(b) A simple contingent claim X written on this asset pays an amount ….. at time t = T.
Show that the arbitrage free value of the claim … at time t = 0 is

By regarding the claim at time t as having time T – t to maturity, deduce from this the arbitrage gree value … of the claim at time t as a fuction of t and St, and verify that ….. **[6 marks]**

(c) Verify that the discounted value …… is a martingale. **[6 marks]**

[Total 20 marks]

QUESTION 6

Suppose that an asset price St obeys the stochastic differential equation

In which the constant r represents the risk free rate of interest, .. is a positive constant, and … refers to a standard Brownian motion under the risk neutral measure Q.

The financial institution Binbets offers the so called cash-or-nothing option on the asset S, with payoff

where the strike K, and the cahs payoff C are positive constants. [The random variable **1{**S_T>K} takes the value 1 if $S_T > K$ and the value 0 otherwise.]

(a) Derive expressions for both the real – world ….. and the risk-nuetral …. Probabilities that the stock price will be higher that K at maturity.
[5 marks]

(b) Hence find the arbitrage free value ….. of the cash-or-nothing option at time 0. [Hint: Remember that ………………] What happens if the strike K becomes very small? **[6 marks]**

(c) By regarding the claim at time t as having time T – t to maturity, deduce from this the time t value …. Of the claim as a function p(t, St) of t and St.
[2 marks]

(d) Determine the delta of this option, and briefly describe how to replicate the option using a portfolio of bond and stock.
[Hint: It will help if you have written down the explicit form of the function p(t,s) in part (c).] **[6 marks]**

(e) By considering a portfolio of two different cash-or-nothing options or otherwise, give a formula for the time t price, ………… of the following contigent claims: