NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

CFE 5302

FACULTY OF COMMERCE DEPARTMENT OF FINANCE

MSc FINANCIAL ENGINEERING

Derivatives Pricing and Applications - CFE 5302

NOV/DEC 2015 EXAMINATION

Time : 3 hours

INSTRUCTIONS:

Candidates should attempt **ALL QUESTIONS.**

This paper carries 100 Marks.

Statistical Tables are attached at the end of the question paper.

QUESTION 1 **[20]**

The forward price of an investment asset providing no income is given by

$$F_0 = S_0 e^{rT},$$

where T is time to maturity, r is the risk-free rate of interest and S_0 is the asset price at time $t = 0$. Consider a four-month forward contract to buy an ounce of gold currently valued at \$1300. Assume that the four-month risk-free rate of interest is 6% per annum.

(a) Obtain the forward price. [3]

(b) What would arbitrageurs do if $F_0 \neq S_0 e^{rT}$? [3]

(c) Suppose f is the value of a long forward contract that has a delivery price of K. The value f is generally given by

$$f = (F_0 - K)e^{-rT}.$$

If the delivery price is \$1320, calculate the value of the long forward contract. [4]

Look at the spot interest rates shown in the following Table:

Year	Spot rate
1	$Y(t; t+1) \equiv r_1 = 0.050$
2	$Y(t; t+2) \equiv r_2 = 0.045$
3	$Y(t; t+3) \equiv r_3 = 0.040$
4	$Y(t; t+4) \equiv r_4 = 0.035$
5	$Y(t; t+5) \equiv r_5 = 0.030$

Suppose that someone told you that the 6-year spot interest rate was 5.50 percent.

(a) Would you believe him or not? Why? [4]

(b) Could you make money if he was right? How? [2]

(c) What is the sensible value for the 6-year spot rate? [4]

QUESTION 2 **[20]**

Consider a simple discrete-time model with $T = 2$ and four states of the world. Suppose $r = 4\%$ and the risky security is as follows:

ω_k	$t = 0$	$t = 1$	$t = 2$
ω_1	$S_0 = 10$	$S_1 = 12$	$S_2 = 13$
ω_2	$S_0 = 10$	$S_1 = 12$	$S_2 = 10$
ω_3	$S_0 = 10$	$S_1 = 8$	$S_2 = 10$
ω_4	$S_0 = 10$	$S_1 = 8$	$S_2 = 7$

(a) Draw the resulting tree diagram. [2]

(b) Find Q_u and Q_d. [3]

(c) Find Q_{uu} and Q_{ud}. [3]

(d) Find Q_{du} and Q_{dd}. [3]

(e) Hence, by showing all the necessary steps, find the discrete time martingale measure Q. [4]

(f) What can you conclude regarding the resulting trading strategy? [2]

QUESTION 3 **[20]**

The Black-Scholes Call and Put formulae are

$$
\begin{aligned}
C &= S_0 N(d_1) - N(d_2) X e^{-rT} \\
P &= N(-d_2) X e^{-rT} - S_0 N(-d_1)
\end{aligned}
$$

where,

$d_1 = [\ln(S_0/X) + (r + 0.5\sigma^2)T]/\sigma\sqrt{T}$,

$d_2 = d_1 - \sigma\sqrt{T}$.

By recalling that

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

and that

$$
\begin{aligned}
\ln \frac{S N'(d_1)}{e^{-r(T-t)} K_c N'(d_2)} &= 0 \\
\Rightarrow S N'(d_1) - e^{-r(T-t)} K_c N'(d_2) &= 0,
\end{aligned}
$$

(a) Deduce an expression for $\rho := \frac{\partial C}{\partial r}$. [8]

(b) Show that the partial derivative $\partial C / \partial K$ (which sadly does not have a Greek name) satisfies [8]

$$\frac{\partial C}{\partial E} = -e^{-r(T-t)} N(d_2).$$

(c) Deduce that $\partial C / \partial K < 0$ and interpret the results. [4]

QUESTION 4 **[20]**

Consider an American option with the following specifications:

• The underlying asset is currently valued at \$120.

• The strike price is set at \$120.

• The risk-free rate is 5% per annum and the riskiness of investing in the underlying is given as 30%. Consider a time partition of 1 year and a maturity of 3 years.

(a) Calculate the down and up movement factors. [2]

(b) Calculate the down and up probabilities. [2]

(c) Hence, calculate all the underlying asset prices. [6]

(d) What is the fair price you should pay to have a **call** option document drafted? Show all the working. [10]

QUESTION 5 **[20]**

As a means to write down an expression for the up/down asset price model used in the binomial method, we define a random variable R_i such that $R_i = 1$ if the asset price goes up from time $(i-1)\delta t$ to $i\delta t$ and $R_i = 0$ if the asset price goes down. Hence, $R_i = 1$ with probability$p$ and $R_i = 0$ with probability $1 - p$. This means that R_i is a Bernoulli random variable with parameter p so that $E(R_i) = p$ and $\text{var}(R_i) = p(1-p)$. After n time increments the asset has undergone $\sum_{i=1}^{n} R_i$ upward movements and $n - \sum_{i=1}^{n} R_i$ downward movements. It can be shown that the asset price $S(n\delta t)$ at time $t = n\delta t$ is given by

$$S(n\delta t) = S_0 u^{\sum_{i=1}^{n} R_i} d^{n-\sum_{i=1}^{n} R_i}. \tag{1}$$

(a) Find an expression for $\log\left(\frac{S(n\delta t)}{S_0}\right)$. [4]

(b) Find expressions for $E\left[\log\left(\frac{S(n\delta t)}{S_0}\right)\right]$ and $\text{var}\left[\log\left(\frac{S(n\delta t)}{S_0}\right)\right]$. [4]

To match the continuous asset price model used in the Black-Scholes analysis, we require the mean of $\log\left(\frac{S(n\delta t)}{S_0}\right)$ to be $(\mu - \frac{1}{2}\sigma^2)n\delta t$ and the variance to be $\sigma^2 n\delta t$. The binomial method works with expected values.

(c) If we impose the risk neutrality assumption $u = r$ on the mean and variance, show that we get the following conditions: [4]

$$p \log u + (1-p) \log d = (r - \frac{1}{2}\sigma^2)\delta t, \tag{2}$$

$$\log\left(\frac{u}{d}\right) = \sigma\sqrt{\frac{\delta t}{p(1-p)}} \tag{3}$$

(d) Show that by setting $p = \frac{1}{2}$, we obtain the following up and down binomial parameters:[4]

$$u = e^{\sigma\sqrt{\delta t}+(r-\frac{1}{2}\sigma^2)\delta t}; \quad d = e^{-\sigma\sqrt{\delta t}+(r-\frac{1}{2}\sigma^2)\delta t} \tag{4}$$

(e) For the parameters u and d above, show that: [4]

$u = 1 + \sigma\sqrt{\delta t} + r\delta t + O(\delta t^{3/2}),\ d = 1 - \sigma\sqrt{\delta t} + r\delta t + O(\delta t^{3/2})$, as $\delta t \to 0$.

END OF EXAMINATION PAPER