NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

FACULTY OF APPLIED SCIENCE

DEPARTMENT OF APPLIED MATHEMATICS

SMA1201: CALCULUS OF SEVERAL VARIABLES

MARCH 2025 EXAMINATION

Time : 3 hours

Candidates should attempt all questions from Section A [**40 MARKS**] and ANY THREE questions in Section B [**60 MARKS**].

SECTION A

A1. Given the function $f(x, y) = \ln(x^2 + y^2 - 16)$,

(a) state the domain of $f(x, y)$, giving your answer in set notation, [2]

(b) sketch in $\mathbb{R}^2$ the domain of $f(x, y)$. [3]

A2. Evaluate the following limits;

(a) $\lim_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2 + y^2}$. [3]

(b) $\lim_{(x,y)\to(1,0)} \dfrac{(x-1)^2 \ln x}{(x-1)^2 + y^2}$. [4]

A3. (a) Given that $z = \dfrac{x}{y}\sin(xy)$, $x = p^2 - 3t$, $y = s^2 - t^2$ and $p = e^{2s}$, find;

(i) $\dfrac{\partial z}{\partial t}$. [3]

(ii) $\dfrac{\partial z}{\partial s}$. [4]

(b) Prove that $\nabla r^n = nr^{n-2}\vec{\mathbf{r}}$, where $r = ||\vec{\mathbf{r}}||$, $n \in \mathbb{R}$ and $\vec{\mathbf{r}} \in \mathbb{R}^3$. [5]

A4. Using polar coordinates, evaluate

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx dy.$$

Hence, deduce that

$$\int_0^\infty e^{-y^2}\, dy = \frac{1}{2}\sqrt{\pi}$$

[7]

A5. (a) Plot the region, R defined as follows:

$$R = \{(x, y) : 1 \leq x \leq e^y \text{ and } 0 \leq y \leq 1\}.$$

[2]

(b) Change the order of integration for the double integral, $\int_0^1 \int_1^{e^y} f(x, y)\, dxdy$. [2]

(c) Find and classify the relative extrema of the function

$$f(x, y) = -x^3 + 4xy - 2y^2 + 1.$$

[5]

SECTION B

B6. (a) Find the total differential of the function $g(x, y) = x^3 y \ln y$. [3]

(b) Find the equation of the tangential plane and the normal line to the surface $z = f(x, y)$ at the point $(1, -2, 12)$, where $f(x, y) = 4x^3y^2 + 2y$. [5]

(c) Determine the directional derivative, $D_{\mathbf{u}}f(x, y)$ in the direction of the unit vector given by $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$, where $\theta = \frac{\pi}{6}$ and $f(x, y) = x^3 - 3xy + 4y^2$.

Hence find $D_{\mathbf{u}}f(1, 2)$. [5]

(d) Find the minimum and maximum values of the function $f(x, y) = 2x - 3y$ subject to the constraint $x^2 + 4y^2 = 25$ using Lagrange multipliers. [7]

B7. For the space curve given by:

$$\mathbf{r}(t) = 2\cos(t)\mathbf{i} + 2\sin(t)\mathbf{j} + t\mathbf{k},$$

(a) Plot the curve traced by $\mathbf{r}(t)$. [3]

(b) Find the velocity and acceleration at $t = \frac{\pi}{4}$. [7]

(c) Show that velocity and acceleration are orthogonal to each other. [4]

(d) Find the bi-normal vector, $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$, where $\vec{T}(t)$ is the unit tangent vector and $\vec{N}(t)$ is the unit normal vector. [6]

B8. (a) The region, D is bounded by the curves $y - x = 3,\ y - x = 0,\ xy = 4$ and $xy = 1$.

(i) Sketch the region. [3]

(ii) Evaluate the integral $\iint_D (x^2 - y^2)\, dA$ by coordinate transformations. [7]

(b) Find the surface area of the portion of the sphere that is within $x^2 + y^2 + z^2 = 9$ and $x^2 + (y - 2)^2 = 4$. [10]

B9. (a) Show that the integral $\oint_C (3x^2y + y)\,dx + (x^3 + x)\,dy$ is independent of path. [3]
Find the scalar potential for this integral and hence evaluate the integral between points $A(1,2)$ and $B(4,5)$. [7]

(b) Use Stokes Theorem to evaluate $\oint_C \mathbf{F}.dr$ where $\mathbf{F} = z^2\vec{i} + y^2\vec{j} + x\vec{k}$ and C is a triangle with vertices at $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. [10]

END OF QUESTION PAPER