# NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA 2104

FACULTY OF APPLIED SCIENCES

**DEPARTMENT OF APPLIED MATHEMATICS**

SMA2104: PARTIAL DIFFERENTIAL EQUATIONS

MARCH 2025 EXAMINATION

Time : 3 hours

Candidates should attempt **ALL** questions from Section A [**40 MARKS**] and **ANY THREE** questions from Section B [**60 MARKS**].

## SECTION A

**ANSWER ALL QUESTIONS [40 MARKS]**

**A1.** Solve the boundary value problem $\dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial x}$; given that $u(x,0) = 4e^{-3x} + 8e^{-5x}$. [6]

**A2.** (a) Distinguish between a regular Sturm-Liouville and a periodic sturm-Liouville problem. [5]

(b) Classify each of the following Sturm-Liouville problems as regular, periodic or singular justifying your answer in each case.

(i) $y'' + \lambda y = 0, \quad y(-\pi) = y(\pi), \quad y'(-\pi) = y'(\pi).$ [2]

(ii) $y'' + y' + \lambda y = 0, \quad y(0) = y(1) = 0.$ [2]

(iii) $((1 - x^2)y')' + \lambda y = 0, \quad [-1; 1].$ [2]

(c) Convert the following differential equation to Sturm-Liouville form ,
$x^2y'' + xy' + (x^2 + \lambda)y = 0.$ [4]

(d) Find the eigenvalues and eigenfunctions for the problem
$y'' + \lambda y = 0, \quad y(0) = y(1) = 0.$ [6]

**A3.** (a) Find the Fourier series of $f(x) = \sin x$, $\frac{-\pi}{2} \leq x < \frac{\pi}{2}$; where $f(x+\pi) = f(x)$. [5]

(b) Find the Fourier half-range sine series of $f(x) = |1-x|$ on $0 \leq x < 2$. [5]

**A4.** Let $F(s)$ denote the Laplace transform of $f(t)$, i.e.

$$F(s) = \int_0^\infty f(t)e^{-st}dt.$$

Use the definition above to find the Laplace transform of $\cosh t$. [3]

## SECTION B

## ANSWER ANY THREE QUESTIONS [60 MARKS]

**B5.** (a) Solve the boundary value problem $\dfrac{\partial^2 u}{\partial x^2} = c^2 \dfrac{\partial^2 u}{\partial t^2}$, $(0 < x < l, t > 0)$
given that $u(0,t) = u(l,t) = 0$, $(t > 0)$;
and $u(x,0) = f(x)$, $\dfrac{\partial u(x,0)}{\partial t} = g(x)$, where $g(x) = x(l-x)$, $(0 < x < l)$. [13]

(b) Find the Laplace transform of $t\cosh t$. [7]

**B6.** (a) Show that for an even function $f(x)$ of period $2l$ which satisifies Dirichlet's conditions,

$$E_n = \int_{-l}^{l} (f(x) - S_n(x))^2 dx$$

is minimised if $a_0 = \dfrac{2}{l}\displaystyle\int_0^l f(x)dx$ and $a_r = \dfrac{2}{l}\displaystyle\int_0^l f(x)\cos(\frac{r\pi x}{l})dx$; where

$$S_n(x) = \frac{a_0}{2} + \sum_{r=1}^{n} a_r \cos\frac{r\pi x}{l} + \sum_{r=1}^{n} b_r \sin\frac{r\pi x}{l}.$$

[7]

(b) (i) Find the Fourier half range cosine series for the function defined by

$$f(x) = \cos x, \quad 0 \leq x < \pi.$$

[6]

(ii) Sketch the function represented by the series over the interval $-3\pi \leq x \leq 3\pi$. [3]

(c) Find the Fourier transform of the function $f(x) = H(x) - 2H(x-1) + H(x-2)$. [4]

**B7.** (a) State Parseval's theorem and use it to find an integral expression for $\pi$ on the function

$$f(x) = H(x+1) - H(x-1).$$

[6]

(b) Use the convolution theorem using

$$f_1(x) = f_2(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

to find $P(\omega)$, the fourier transform of $p(x) = f_1(x).f_2(x)$. [6]

(c) Solve $\dfrac{d^2y}{dt^2} + 8y = f(t)$ where $f(t) = |t|; \quad -1 \leq t < 1; \quad f(t+2) = f(t)$. [8]

**B8.** (a) Show that the eigenfunctions of the Sturm-Liouville problem $y'' + \lambda y = 0$, $3y(1) + y'(1) = 0$, $y(0) = 0$ are given by $y = c \sin \alpha x$ where $\alpha$ is chosen such that $\tan \alpha = \frac{-\alpha}{3}$. [10]

(b) Use the method of seperation of variables to solve the laplace equation
$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \quad (0 < x < 1), (0 < y < 1),$
given that $u(x,0) = u(x,1) = 0; \quad 0 < x < 1;$
$u(0,y) = 0$ and $u(1,y) = 100; \qquad 0 < y < 1$ . [10]

**END OF QUESTION PAPER**