NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA3116

FACULTY OF APPLIED SCIENCE

DEPARTMENT OF APPLIED MATHEMATICS

SMA3116: ENGINEERING MATHEMATICS IV

DECEMBER 2024 EXAMINATION

Time : 3 hours

This paper contains **TWO** sections. Attempt **ALL** questions from Section A and any **THREE** questions from Section B

SECTION A: Answer ALL questions in this section [40].

Attempt **ALL** questions from this section [**40 MARKS**]

A1. Determine the Lagrange form of the cubic interpolating function for the given data:

x	1	2	4	3
f(x)	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{3}$

and use the result to estimate $f(0.75)$. [5]

A2. (a) Show that the approximation to the Second derivative of a function $f(x)$ is given by

$$f''(x_0) \approx \frac{1}{h^2}\left[f(x_0 - h) - 2f(x_0) + f(x_0 + h)\right]$$

[6]

(b) Hence or otherwise, determine the approximation to the second derivative of

$$f(x) = xe^x$$

at $x = 2.0$, that is, $(f''(2.0))$ with $h = 0.1$. [4]

A3. Use Simpson's rule with $n = 8$ to approximate the integral

$$\int_0^{\pi/2} cos(x)dx$$

[5]

A4.

Obtain the Euler approximation to the solution of the initial value problem

$$y' = 4y, \quad y(0) = 1$$

on [0,0.15], using $h = 0.05$. Compare your results with the analytic solution. [5]

A5.

Solve the following initial value problem

$$y'(x) = x + y^2, \quad y(0) = 1$$

(a) by the Euler-Cauchy method;

(b) by the Fourth Order Runge-Kutta method.

using $h = 0.1$ to estimate the solution at $x = 0.2$. Compare the results. [5,6]

A6. Determine and classify all the singular points for the finite values of x of the differential equation

$$(x^2 - 4)^2 y'' + (x - 2)y' + 2y = 0$$

[4]

SECTION B: Answer THREE questions in this section [60].

Answer any **THREE** questions from this section **[60 MARKS]**

B7. (a) Use the LU-decomposition method to solve the following system of linear equations:

$$\begin{aligned} 2x + 3y + z &= 5 \\ 4x + 7y + 2z &= 10 \\ 6x + 18y + 5z &= 15 \end{aligned}$$

[7]

(b) Apply four iterations of the Gauss-Seidel iteration method, using $x_0 = y_0 = z_0 = 0$ as initialization, to solve the following system:

$$\begin{aligned} 4x - y + z &= 7 \\ -x + 3y - z &= 4 \\ x + y + 5z &= 6 \end{aligned}$$

[5]

(c) Consider the set of equations:

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{aligned}$$

Suppose the true solutions of these equations x, y and z and the iterative values after r steps of the Jacobi process are x_r, y_r and z_r where

$$x_r = x + \alpha_r \quad y_r = y + \beta_r \quad z_r = z + \gamma_r$$

Let $E_r = \max\{|\alpha_r|, |\beta_r|, |\gamma_r|\}$, i.e. the largest error after r steps.

(i) Prove that

$$\begin{aligned} |\alpha_{r+1}| &\leq |\frac{a_{12}}{a_{11}}||\beta_r| + |\frac{a_{13}}{a_{11}}||\gamma_r| \\ |\beta_{r+1}| &\leq |\frac{a_{21}}{a_{22}}||\alpha_r| + |\frac{a_{23}}{a_{22}}||\gamma_r| \\ |\gamma_{r+1}| &\leq |\frac{a_{31}}{a_{33}}||\alpha_r| + |\frac{a_{32}}{a_{33}}||\beta_r| \end{aligned}$$

(ii) Comment on the convegence scheme if

$$\begin{aligned} |a_{11}| &> |a_{12}| + |a_{13}| \\ |a_{22}| &> |a_{21}| + |a_{23}| \\ |a_{33}| &> |a_{31}| + |a_{32}| \end{aligned}$$

[8]

B8. (a) Consider the heat equation

$$\frac{\partial}{\partial t}u(x,t) = \frac{1}{\pi^2}\frac{\partial^2}{\partial x^2}u(x,t) \quad 0 < x < 1, \quad 0 < t$$

with boundary conditions $u(0,t) = 0, u(1,t) = 0,\ t > 0$ and initial conditions $u(x,0) = sin(\pi x)$, $0 < x < 1$. Determine $u(x,t)$. [10]

(b) Assume an elastic string with fixed ends is plucked like a guitar string. The governing equation for $u(x,t)$, the position of the string from its equilibrium position, is the wave equation

$$\frac{\partial^2}{\partial t^2}u(x,t) = \frac{1}{4}\frac{\partial^2}{\partial x^2}u(x,t) \quad 0 < x < \pi, \quad 0 < t$$

with boundary conditions $u(0,t) = 0, u(\pi,t) = 0,\ t > 0$ and initial conditions $u(x,0) = sin(2x)$, $0 < x < \pi$ and $\frac{\partial}{\partial t}u(x,0) = 0$, $0 < x < \pi$. Determine $u(x,t)$. [10]

B9. (a) (i) Use the Bisection method to find the solution x^* of

$$x = e^{-x}$$

correct to an accuracy of 10^{-2}, where $x^* \in [0,1]$.
How many iterations are needed to achieve an accuracy of 10^{-9} [5]

(ii) Apply the Secant method to approximate the solution x^* of

$$2 = e^{x}$$

correct to an accuracy of 10^{-4}, where $x^* \in [0,1]$ [5]

(b) Suppose that y_r is $k-$digit rounding approximation to y. Show that

$$\left|\frac{y - y_r}{y}\right| \leq \frac{1}{2} \times 10^{-k+1}$$

[10]

B10. (a) Solve the Euler-Cauchy equation

$$x^2y'' + 4xy' + 2y = 0$$

by assuming Frobenius series solution. [10]

(b) Given the following Euler-Cauchy equation

$$y'' - y = 0$$

Show that the solution of the differential equation is given by

$$y(x) = a_0\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} + a_1\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$$

[10]

END OF QUESTION PAPER