NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS

SMA4162: NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS

AUGUST 2024: SUPPLEMENTARY EXAMINATION

Time : 3 hours

Candidates should attempt **ALL** questions from **Section A** (40 marks) and **ANY THREE** questions from **Section B** (20 marks each).

SECTION A

Answer ALL questions [**40 MARKS**]

A1. Use Lipschitz theorem to establish if the following initial-value problems are well-posed;

(a) $y' = 1 + t\sin(ty), 0 \le t \le 2, y(0) = 0,$ [4]

(b) $y' = -ty + \dfrac{4t}{y}, 0 \le t \le 1, y(0) = 1.$ [4]

A2. (a) Distinguish between single-step method and multi-step methods. [4]

(b) State the advantages and disavantages of the methods described in (a) above. [3]

A3. For the initial-value problem
$y' = te^{3t} - 2y, 0 \le t \le 1, y(0) = 0$ with $h = 0.25$, use;

(a) (i) Euler's method to solve the problem. [5]

(ii) Taylor's Fourth order. [8]

(b) Compare the result with the exact values given by $y(t) = \dfrac{e^{-2t}}{25}(1 - e^{5t} + 5te^{5t})$ and draw conclusions. [2]

A4. Use the Forward-difference method to solve the one dimensional heat-equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}, 0 \le x \le 1, t > 0$$

with boundary conditions

$U(0,t) = U(1,t) = 0$

and initial condition

$U(x,0) = \sin(\pi x), 0 \le x \le 1$

to compare the result at $t = 0.0005$ with the exact solution $U(x,t) = e^{-\pi^2 t}\sin(\pi x)$.
Use the step sizes $x_{i+1} - x_i = h = 0.1$ and $t_{j+1} - t_j = k = 0.0005$.   [10]

SECTION B

Answer ANY THREE questions [60 MARKS]

B5. (a) State the advantages and disadvantages of using explicit and implicit methods. [3]

(b) Explain how the predictor-corrector method works.   [3]

(c) Use the fourth order Adams-Bashforth explicit method as the predictor

$$\omega_{p_{i+1}} = \omega_i + \frac{h}{24}[55f(t_i,\omega_i) - 59f(t_{i-1},\omega_{i-1}) + 37f(t_{i-2},\omega_{i-2}) - 9f(t_{t-3},\omega_{i-3})]$$

and the fourth order Adams-Moulton implicit method as the corrector

$$\omega_{i+1} = \omega_i + \frac{h}{24}[9f(t_{i+1},\omega_{p_{i+1}}) + 19f(t_i,\omega_i) - 5f(t_{i-1},\omega_{i-1}) + f(t_{i-2},\omega_{i-2})]$$

to find an approximation to the initial-value problem
$y' = t + y, 0 \le t \le 1, y(0) = 3$ using $h = 0.2$.
Use the exact values given by $y(t) = 4e^t - t - 1$ to estimate the first three values. [6]

(d) Repeat the problem above using a step size $h = 0.1$ and comment on the effect of the step size.   [8]

B6. (a) Derive the Crank-Nicolson formula.   [10]

(b) Determine the steady-state heat distribution in a thin right-angled triangular metal sheet of $1m \times 1m$ using $m = n = 5$. The diagonal side is kept at 0^0C and heat on the other boundaries increases linearly from 0^0C to 100^0C at the corner where the two sides meet.
NB: heat distribution is modelled by the Lapalce's equation $\nabla^2 U = 0$. Do not solve the resulting $m \times m$ matrix.   [10]

B7. (a) Define the following terms in the context of numerical methods

(i) consistence,   [2]

(ii) convergence and   [2]

(iii) stability.   [2]

(b) Use Runge-Kutta fourth order method with $h = 0.1$ to solve the second-order initial-value problem
$y'' - 2y' + 2y = e^{2t}\sin(t)$ for $0 \le t \le 1$ with initial conditions $y(0) = -0.4$ and $y' = -0.6$.
Approximate the values of $y(0.1)$, $y(0.2)$ and $y(0.3)$. [14]

B8. (a) Determine if the following boundary value problems have unique solutions;

(i) $y'' = y' + 2y + \cos(x), 0 \le x \le \dfrac{\pi}{2}, y(0) = -0.3, y(\dfrac{\pi}{2}) = -0.1,$ [3]

(ii) $y'' = -3y' + 2y + 2x + 3, 0 \le x \le 1, y(0) = 2, y(1) = 1.$ [3]

(b) Use the linear shooting method to solve the boundary-value problem
$y'' = 4(y - x), 0 \le x \le 1, y(0) = 0, y(1) = 2$. Use the Euler's method with $h = 0.2$ and $h = 0.25$. Compare the result with the exact solution
$$y(x) = \frac{e^2}{e^4 - 1}(e^{2x} - e^{-2x}) + x$$
and discuss the effect of the step size. [14]

END OF QUESTION PAPER