

NATIONAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

SMA4162

FACULTY OF APPLIED SCIENCE

DEPARTMENT OF APPLIED MATHEMATICS

SMA 4162: NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS

DECEMBER 2024: EXAMINATION

Time : 3 hours

Candidates should attempt **ALL** questions from **Section A**(40 MARKS) and **ANY THREE** questions from **Section B** (60 MARKS).

SECTION A (40 MARKS)

Answer **ALL** questions(40 MARKS)

A1. (a) State the Lipschitz condition. [3]

(b) Use Lipschitz theorem to establish if the following initial-value problems are well-posed;

(i) $y' = 1 + t \sin(ty)$, $0 \leq t \leq 2$, $y(0) = 0$, [4]

(ii) $y' = -ty + \frac{4t}{y}$, $0 \leq t \leq 1$, $y(0) = 1$. [4]

A2. (a) Use the fourth order Taylor's series method to solve the problem $y' = y - t^2 + 1$, $y(0) = 0.5$ using a step size $h = 0.4$ for $0 \leq t \leq 2$. [8]

(b) Use the Euler 's method to solve the initial value problem $y' = t + y$, $y(0) = 0$ using $h = 0.2$ for $0 \leq t \leq 1$. [5]

A3. (a) Use the Forward-difference method to solve the one dimensional heat-equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} , 0 \leq x \leq 1 , t > 0$$

with boundary conditions $U(0, t) = U(1, t) = 0$ and initial conditions $U(x, 0) = \sin(\pi x)$, $0 \leq x \leq 1$ to compare the result at $t = 0.0005$ with the exact solution $U(x, t) = e^{-\pi^2 t} \sin(\pi x)$. Use the step sizes $x_{i+1} - x_i = h = 0.1$ and $t_{j+1} - t_j = k = 0.0005$. [8]

- A4. (a) Write a program in Matlab to solve the IVP $y'(t) = t + y$, $0 \leq t \leq 1$, $y(0) = 0$, the exact solution to the IVP is $y(t) = e^t - t - 1$. [8]

SECTION B

Answer ANY THREE questions(60 MARKS)

- B5. (a) Derive the Crank-Nicholson formula. [10]
- (b) Determine the steady-state heat distribution in a thin rectangular metal sheet of $0.5m \times 0.5m$ using $m = n = 5$. The sides $u(x, 0) = u(0, y) = 0$ are kept at $0^\circ C$ and heat on the other boundaries increases linearly from $0^\circ C$ to $100^\circ C$ at the corner where the two sides meet.
NB: heat distribution is modelled by the Laplace's equation $\nabla^2 U = 0$. Do not solve the resulting $m \times m$ matrix. [10]
- B6. (a) Determine if the following boundary value problems have unique solutions;
- (i) $y'' = -3y' + 2y + 2x + 3$, $0 \leq x \leq 1$, $y(0) = 2$, $y(1) = 1$. [3]
- (ii) $y'' + e^{-xy} + \sin y' = 0$, $1 \leq x \leq 2$, $y(1) = y(2) = 0$. [3]
- (b) Use the linear shooting method to solve the boundary-value problem $y'' - 3y' + 2y = x$, $y(0) = 1$, $y(1) = 2$. Use the Euler's method with $h = 0.25$ to solve the problem. [14]
- B7. (a) State the advantages and disadvantages of using explicit and implicit methods. [3]
- (b) Explain how the predictor-corrector method works. [3]
- (c) Use the fourth order Adams-Bashforth explicit method as the predictor $\omega_{p_{i+1}} = \omega_i + \frac{h}{24}[55f(t_i, \omega_i) - 59f(t_{i-1}, \omega_{i-1}) + 37f(t_{i-2}, \omega_{i-2}) - 9f(t_{i-3}, \omega_{i-3})]$ and the fourth order Adams-Moulton implicit method as the corrector $\omega_{i+1} = \omega_i + \frac{h}{24}[9f(t_{i+1}, \omega_{p_{i+1}}) + 19f(t_i, \omega_i) - 5f(t_{i-1}, \omega_{i-1}) + f(t_{i-2}, \omega_{i-2})]$ to find an approximation to the initial-value problem $y' = t + y$, $0 \leq t \leq 1$, $y(0) = 3$ using $h = 0.2$.
Use the exact values given by $y(t) = 4e^t - t - 1$ to estimate the first three values. [14]

B8. (a) Define the following terms in the context of numerical methods

(i) consistence, [2]

(ii) convergence and [2]

(iii) stability. [2]

(b) Use the finite element method (FEM) using step size $h = 0.25$ to solve the boundary value problem $-u'' = x$, $u(0) = 0$, $u'(1) = 0$.

NB: DO NOT SOLVE THE RESULTING $M \times M$ MATRIX.. [14]